Bounds for the price of discretely sampled arithmetic Asian options

M. Vanmaele[*], G. Deelstra[*], J. Liinev[*], J. Dhaene[†], M.J. Goovaerts[†]

Version: September 17, 2002

[*]Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281, S9, 9000 Gent, Belgium, Michele.Vanmaele@rug.ac.be, Griselda.Deelstra@rug.ac.be, Jan.Liinev@rug.ac.be, Tel.: +32 9 2644895

[†]Department of Applied Economics, Catholic University Leuven, Naamsestraat 69, 3000 Leuven, Belgium, Jan.Dhaene@econ.kuleuven.ac.be, Marc.Goovaerts@econ.kuleuven.ac.be, Tel. +32 16 326750

Bounds for the price of discretely sampled arithmetic Asian options

Abstract

In this paper the pricing of discretely sampled European-style arithmetic Asian options with fixed and floating strike is studied by deriving accurate lower and upper bounds. For a lower bound, we adapt the idea of Rogers and Shi (1995) and we use in addition results based on comonotonic risks. For an upper bound we first use a general technique for deriving upper (and lower) bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas, Dhaene and Goovaerts (2000) and we further use the ideas of Rogers and Shi (1995) and Nielsen and Sandmann (2002). We obtain analytical and easily computable bounds. We also study the hedging problem. Several sets of numerical results are included.

I Introduction

In this paper the pricing of discretely sampled European-style arithmetic Asian options with fixed and floating strike is studied. Asian options are path-dependent contingent claims with pay-offs that depend on the average of the underlying asset price over some prespecified period of time, often a low number of trading days in the discrete case. Such contracts form an attractive specification for thinly traded asset markets where price manipulation on or near a maturity date is possible. In markets where prices are prone to periods of extreme volatility the averaging performs a smoothing operation. For buyers as well as for writers, an Asian option is a useful hedging instrument. These Asian options provide for the buyer a cost efficient way of hedging cash or asset flows over extended periods, e.g., for foreign exchange, interest rate, or commodities like oil or gold. For the writer of an Asian option, the advantages include more manageable hedge ratios and the ability to unwind his position more gracefully at the end.

Asian options can also be part of complex financial contracts and strategies, like retirement plans or catastrophe insurance derivatives. Indeed, as explained in Nielsen and Sandmann (2002), a typical investment plan of a retirement scheme could include fixed periodic payments invested in a specified risky asset. An Asian option on the average return can be used to guarantee a minimum rate of return on the periodic payments. On the other hand, Cat-calls are catastrophic risk options which include Asian options on the average of an underlying index (see Geman (1994)).

Within the Black & Scholes (1973) model, no closed form solutions are available for Asian options involving the discretely sampled arithmetic average. For unlike options on geometric average, the density function for the arithmetic average is not lognormal and has no explicit representation. A variety of methods for the European case and especially continuous fixed strike options have been developed while only a few papers deal with the more practical case of discrete arithmetic averaging. A partial list includes (for references see for example Klassen (2001) and Večeř (2001)): Monte Carlo or quasi-Monte Carlo methods, exact expressions involving Laplace transforms or an infinite sum over recursively defined integrals, convolution methods using the fast Fourier transform, analytic approximations based on moment matching or conditioning on some average, a number of PDE methods, tree methods.

An accurate lower and upper bound in the case of continuous averaging was obtained by the method of conditioning in Rogers and Shi (1995). We adapt this idea to the case of discrete averaging and use in addition results based on comonotonic risks (see Kaas, Dhaene and Goovaerts (2000) and Dhaene et al. (2002)). This approach leads to an accurate, analytical, easily computable lower bound for the price of an Asian option. For an upper bound we follow different approaches, one that is based on a general technique for deriving upper (and lower) bounds for stop-loss premiums of sums of dependent random variables, as explained in Kaas, Dhaene and Goovaerts (2000), another that follows again the ideas of Rogers and Shi (1995) and Nielsen and Sandmann (2002), and a third one that combines the two methods. They all lead to analytical, computable upper bounds. We compare all approaches and compare our results to those of Jacques (1996), who approximates the distribution of the arithmetic average by a more tractable one.
As in Nielsen and Sandmann (2002), we then study the hedging of Asian options by calculating the Delta, Gamma and Vega of the lower and upper bounds.

An arithmetic Asian European-style call option with exercise date T, n averaging dates and fixed exercise price K, generates at T a pay-off

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S(T-i) - K\right)_+ ,$$

where $x_+ = \max\{x, 0\}$ and $S(T-i)$ is the price of a risky asset at time $T-i$, $i = 0, \ldots, n-1$. The price of the call option at current time $t = 0$ is given by

$$AC(n, K, T) = \frac{e^{-rT}}{n} E^Q\left[\left(\sum_{i=0}^{n-1} S(T-i) - nK\right)_+\right] \tag{1}$$

under a martingale measure Q and with r the risk-neutral interest rate. Throughout the paper we consider 'forward starting' Asian options which means that at the current time 0, the averaging has not yet started and that the n variables $S(T-n+1), \ldots, S(T)$ are random. This case states in contrast with the case that $T-n+1 \le 0$ where only the prices $S(1), \ldots, S(T)$ remain random. In literature, this Asian option is called 'in progress'. Most papers treat only standard Asian options which is the case of $T = n-1$. Note however that our results for forward starting Asian options can immediately be translated to results for Asian options in progress.
Assuming a Black & Scholes setting, the random variables $S(T-i)/S(0)$ are lognormally distributed under the unique risk-neutral measure Q with parameters $(r-\sigma^2/2)(T-i)$ and $\sigma^2(T-i)$, when σ is the volatility of the underlying risky asset. Therefore we do not have an explicit analytical expression for the distribution of the average $\frac{1}{n}\sum_{i=0}^{n-1} S(T-i)$ and determining the price of the Asian option is a complicated task. From (1) it is seen that the problem of pricing arithmetic Asian options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent risks. Hence we can apply the results on comonotonic upper and lower bounds for stop-loss premiums, which have been summarized in Section II.
Simon, Goovaerts and Dhaene (2000) derived and computed in a general framework an analytical expression for the so-called 'comonotonic upper bound', which is in fact the smallest linear combination of prices of European call options that bounds the price of an European-style Asian option from above. Nielsen and Sandmann (2002) studied both upper and lower bounds for an European-style arithmetic Asian option in the Black & Scholes setting. In particular, they derive

the Simon, Goovaerts and Dhaene upper bound using Lagrange optimization. Nielsen and Sandmann (2002) also apply the Rogers & Shi reasoning in the arithmetic case by using one specific standardized normally distributed conditioning variable.
Independently, we derive more general lower and Rogers & Shi upper bounds in the sense that we allow for other normally distributed conditioning variables.

Next, we improve the Rogers & Shi upper bounds and obtain another so-called partially exact/comonotonic upper bound which consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part. This idea of decomposing the calculations in an exact part and an approximating part goes at least back to Curran (1994). This last upper bound generalizes an analogous bound of Nielsen and Sandmann and provides a closed-form expression for it.

The procedures in this paper can also be used to derive directly upper and lower bounds for the price of arithmetic Asian put options. We price here, instead, the fixed strike put options by means of the put-call parity.

An arithmetic Asian European-style put option with exercise date T, n averaging dates ($n \leq T+1$) and floating exercise price with percentage β, generates at T a pay-off

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S(T-i) - \beta S(T)\right)_+ .$$

By using a change of numeraire, we can evaluate these financial instruments as well. In independent work, Henderson and Wojakowski (2002) use the same change of numeraire to obtain symmetry results between forward starting floating and fixed strike Asian options in case of continuous sampling. We show that their results can be extended to discretely sampling and we give also bounds for the Asian floating options in progress. In fact, Henderson and Wojakowski (2002) consider the case of a continuous dividend yield δ. This case can also be easily dealt with in our context, by replacing the interest rate r by $r-\delta$. We only treat the continuous dividend yield δ explicitly in our generalization of the Henderson and Wojakowski symmetry results to the arithmetic discrete sampling case.

The paper is composed as follows. Section II recalls from Kaas et al. (2000) procedures for obtaining the lower and upper bounds for prices by using the notion of comonotonicity. Section III applies these procedures in the case of a sum of lognormal variables. Section IV provides bounds for the fixed strike Asian options in the Black & Scholes setting, first by concentrating on the comonotonicity and then by applying the Rogers and Shi approach to carefully chosen conditioning variables. We also provide upper bounds by generalizing the Nielsen and Sandmann idea and by combining it with the notion of comonotonicity. Several sets of numerical results are given and the different bounds are discussed. We further derive hedging formulae for the lower and upper bounds. Section V treats the floating strike Asian options in the Black & Scholes setting. Section VI concludes the paper.

II Some theoretical results

In this section, we recall from Dhaene et al. (2002) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $\mathbb{S}$ of dependent random variables by using the notion of comonotonicity. A random vector $(X_1^c, \ldots, X_n^c)$ is *comonotonic* if each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

In both financial and actuarial context one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=1}^n X_i$ where the terms X_i are not mutually independent, but the multivariate distribution function of the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables X_i. In such cases, to be able to make decisions it may be helpful to find the dependence structure for the random vector $(X_1, \ldots, X_n)$ producing the least favourable aggregate claims $\mathbb{S}$ with given marginals. Therefore, given the marginal distributions of the terms in a random variable $\mathbb{S} = \sum_{i=1}^n X_i$, we shall look for the joint distribution with a smaller resp. larger sum, in the convex order sense. In short, the sum $\mathbb{S}$ is bounded below and above in convex order ($\preceq_{\mathrm{cx}}$) by sums of comonotonic variables:

$$\mathbb{S}^\ell \preceq_{\mathrm{cx}} \mathbb{S} \preceq_{\mathrm{cx}} \mathbb{S}^u \preceq_{\mathrm{cx}} \mathbb{S}^c,$$

which implies by definition of convex order that

$$E[(\mathbb{S}^\ell - d)_+] \leq E[(\mathbb{S} - d)_+] \leq E[(\mathbb{S}^u - d)_+] \leq E[(\mathbb{S}^c - d)_+]$$

for all d in $\mathbb{R}^+$, while $E[\mathbb{S}^\ell] = E[\mathbb{S}] = E[\mathbb{S}^u] = E[\mathbb{S}^c]$.

A. Comonotonic upper bound

As proven in Dhaene et al. (2002), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $\mathbb{S}^c = X_1^c + X_2^c + \cdots + X_n^c$ with

$$\mathbb{S}^c \overset{d}{=} \sum_{i=1}^n F_{X_i}^{-1}(U),$$

where the usual inverse of a distribution function, which is the non-decreasing and left-continuous function $F_X^{-1}(p)$ is defined by

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0,1],$$

with $\inf \emptyset = +\infty$ by convention.

Kaas et al. (2000) have proved that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions. Therefore, given the inverse functions $F_{X_i}^{-1}$, the cumulative density function (hereafter denoted by cdf) of $\mathbb{S}^c = X_1^c + X_2^c + \cdots + X_n^c$ can be determined as follows:

$$\begin{aligned} F_{\mathbb{S}^c}(x) &= \sup \{p \in [0,1] \mid F_{\mathbb{S}^c}(x) \geq p\} = \sup \left\{p \in [0,1] \mid F_{\mathbb{S}^c}^{-1}(p) \leq x\right\} \\ &= \sup \left\{ p \in [0,1] \mid \sum_{i=1}^n F_{X_i}^{-1}(p) \leq x \right\}. \end{aligned} \tag{2}$$

Hence, in case of strictly increasing and continuous marginals, the cdf $F_{\mathbb{S}^c}(x)$ is uniquely determined by

(3) $$\sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{\mathbb{S}^c}\left(x\right)\right) = x, \qquad F_{\mathbb{S}^c}^{-1}(0) < x < \ F_{\mathbb{S}^c}^{-1}(1).$$

Hereafter we restrict ourselves to this case of strictly increasing and continuous marginals.

In the following theorem Dhaene et al. (2002) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

Theorem 1. *The stop-loss premiums of the sum* $\mathbb{S}^c$ *of the components of the comonotonic random vector* $(X_1^c, X_2^c, \ldots, X_n^c)$ *are given by*

$$E\left[(\mathbb{S}^c - d)_+\right] = \sum_{i=1}^{n} E\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}\left(d\right)\right)\right)_+\right], \qquad (F_{\mathbb{S}^c}^{-1}(0) < d < \ F_{\mathbb{S}^c}^{-1}(1)).$$

If the only information available concerning the multivariate distribution function of the random vector $(X_1, \ldots, X_n)$ are the marginal distribution functions of the X_i, then the distribution function of $\mathbb{S}^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U)$ is a prudent choice for approximating the unknown distribution function of $\mathbb{S} = X_1 + \cdots + X_n$. It is a supremum in terms of convex order. It is the best upper bound that can be derived under the given conditions.

B. Improved comonotonic upper bound

Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable Λ with a given distribution function, such that we know the conditional cumulative distribution functions, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. In fact, Kaas et al. (2000) define the improved comonotonic upper bound $\mathbb{S}^u$ as

$$\mathbb{S}^u = F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \cdots + F_{X_n|\Lambda}^{-1}(U),$$

where $F_{X_i|\Lambda}^{-1}(U)$ is the notation for the random variable $f_i(U, \Lambda)$, with the function f_i defined by $f_i(u, \lambda) = F_{X_i|\Lambda=\lambda}^{-1}(u)$. In order to obtain the distribution function of $\mathbb{S}^u$, observe that given the event $\Lambda = \lambda$, the random variable $\mathbb{S}^u$ is a sum of comonotonic random variables. Hence,

$$F_{\mathbb{S}^u|\Lambda=\lambda}^{-1}(p) = \sum_{i=1}^{n} F_{X_i|\Lambda=\lambda}^{-1}(p), \quad p \in [0, 1]\,.$$

Given $\Lambda = \lambda$, the cdf of $\mathbb{S}^u$ is defined by

$$F_{\mathbb{S}^u|\Lambda=\lambda}(x) = \sup\left\{p \in [0, 1] \ \middle|\ \sum_{i=1}^{n} F_{X_i|\Lambda=\lambda}^{-1}(p) \leq x\right\}.$$

The cdf of $\mathbb{S}^u$ then follows from

$$F_{\mathbb{S}^u}(x) = \int_{-\infty}^{+\infty} F_{\mathbb{S}^u|\Lambda=\lambda}(x)\, dF_\Lambda(\lambda).$$

If the marginal cdfs $F_{X_i|\Lambda=\lambda}$ are strictly increasing and continuous, then $F_{\mathbb{S}^u|\Lambda=\lambda}(x)$ is a solution to

$$\text{(4)} \qquad \sum_{i=1}^{n} F^{-1}_{X_i \mid \Lambda=\lambda}\left(F_{\mathbb{S}^u \mid \Lambda=\lambda}(x)\right) = x, \qquad x \in \left(F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(1)\right).$$

In this case, we also find that for any $d \in \left(F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(1)\right)$:

$$E\left[(\mathbb{S}^u - d)_+ \mid \Lambda = \lambda\right] = \sum_{i=1}^{n} E\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}\left(F_{\mathbb{S}^u|\Lambda=\lambda}(d)\right)\right)_+ \mid \Lambda = \lambda\right],$$

from which the stop-loss premium at retention d of $\mathbb{S}^u$ can be determined by integration with respect to λ.

C. Lower bound

Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. We assume as in the previous section that there exists some random variable Λ with a given distribution function, such that we know the conditional cdfs, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. This random variable Λ, however, should not be the same as in case of the upper bound. We recall from Kaas et al. (2000) how to obtain a lower bound, in the sense of convex order, for $\mathbb{S} = X_1 + X_2 + \cdots + X_n$ by conditioning on this random variable. We remark that this idea also can be found in Rogers and Shi (1995) for the continuous case.

Let us denote the conditional expectation by $\mathbb{S}^\ell$:

$$\mathbb{S}^\ell = E\left[\mathbb{S} \mid \Lambda\right].$$

Let us further assume that the random variable Λ is such that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing and continuous functions of Λ. The quantiles of the lower bound $\mathbb{S}^\ell$ then follow from

$$F^{-1}_{\mathbb{S}^\ell}(p) = \sum_{i=1}^{n} F^{-1}_{E[X_i|\Lambda]}(p) = \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_\Lambda(p)\right], \quad p \in [0,1],$$

and the cdf of $\mathbb{S}^\ell$ is according to (2) given by

$$F_{\mathbb{S}^\ell}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_\Lambda(p)\right] \leq x\right\}.$$

If we now additionally assume that the cdfs of the random variables $E\left[X_i \mid \Lambda\right]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell$ is also strictly increasing and continuous, and we get for all $x \in \left(F^{-1}_{\mathbb{S}^\ell}(0), F^{-1}_{\mathbb{S}^\ell}(1)\right)$,

$$\text{(5)} \qquad \sum_{i=1}^{n} F^{-1}_{E[X_i|\Lambda]}\left(F_{\mathbb{S}^\ell}(x)\right) = x \quad \Leftrightarrow \quad \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_\Lambda\left(F_{\mathbb{S}^\ell}(x)\right)\right] = x,$$

which unambiguously determines the cdf of the convex order lower bound $\mathbb{S}^\ell$ for $\mathbb{S}$. Using Theorem 1, the stop-loss premiums of $\mathbb{S}^\ell$ can be computed as:

$$E\left[\left(\mathbb{S}^\ell - d\right)_+\right] = \sum_{i=1}^{n} E\left[\left(E\left[X_i \mid \Lambda\right] - E\left[X_i \mid \Lambda = F_\Lambda^{-1}\left(F_{\mathbb{S}^\ell}(d)\right)\right]\right)_+\right],$$

which holds for all retentions $d \in \left(F_{\mathbb{S}^\ell}^{-1}(0),\ F_{\mathbb{S}^\ell}^{-1}(1)\right)$.

So far, we considered the case that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing functions of Λ. The case where all $E\left[X_i \mid \Lambda\right]$ are non-increasing and continuous functions of Λ also leads to a comonotonic vector $\left(E\left[X_1 \mid \Lambda\right], E\left[X_2 \mid \Lambda\right], \ldots, E\left[X_n \mid \Lambda\right]\right)$, and can be treated in a similar way.

III Sums of lognormal variables

In this section, we study upper and lower bounds for $E\left[(\mathbb{S} - d)_+\right]$ where $\mathbb{S}$ is a linear combination of lognormal variables. Let us denote

(6)
$$\mathbb{S} = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \alpha_i\, e^{Y_i},$$

with Y_i a normally distributed random variable with mean $E\left[Y_i\right]$ and variance $\sigma_{Y_i}^2$. In this case the stop-loss premium with some retention d_i, namely $E\left[(X_i - d_i)_+\right]$, is well-known since $\ln\left(\operatorname{sign}(\alpha_i)\, X_i\right) \sim N\left(\mu_i, \sigma_i^2\right)$ with

$$\mu_i = \ln|\alpha_i| + E[Y_i], \quad \text{and} \quad \sigma_i = \sigma_{Y_i},$$

and equals for $\alpha_i d_i > 0$

(7)
$$E[(X_i - d_i)_+] = \operatorname{sign}(\alpha_i)\, e^{\mu_i + \frac{\sigma_i^2}{2}}\ \Phi(\operatorname{sign}(\alpha_i)\, d_{i,1}) - d_i\ \Phi(\operatorname{sign}(\alpha_i)\, d_{i,2}),$$

where $d_{i,1}$ and $d_{i,2}$ are determined by

(8)
$$d_{i,1} = \frac{\mu_i + \sigma_i^2 - \ln|d_i|}{\sigma_i}, \quad d_{i,2} = d_{i,1} - \sigma_i.$$

The cases $\alpha_i d_i < 0$ are trivial.

We now consider a normally distributed random variable Λ and we slightly generalize Theorem 1 of Dhaene et al. (2002) to our more general settings.

Theorem 2. *Let $\mathbb{S}$ be given by* (6) *and consider a normally distributed random variable Λ which is positively correlated to all Y_i in $\mathbb{S}$ and such that (Y_i, Λ) is bivariate normally distributed for all i. Then the distributions of the comonotonic upper bound $\mathbb{S}^c$, the improved comonotonic upper* *bound $\mathbb{S}^u$ and the lower bound $\mathbb{S}^\ell$ are given by*

(9) $$\mathbb{S}^c = \sum_{i=1}^n F_{X_i}^{-1}(U) = \sum_{i=1}^n \alpha_i \, e^{E[Y_i]+\text{sign}(\alpha_i)\sigma_{Y_i}\Phi^{-1}(U)},$$

(10) $$\mathbb{S}^u = \sum_{i=1}^n F_{X_i|\Lambda}^{-1}(U) = \sum_{i=1}^n \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(V)+\text{sign}(\alpha_i)\sqrt{1-r_i^2}\,\sigma_{Y_i}\Phi^{-1}(U)},$$

(11) $$\mathbb{S}^\ell = \sum_{i=1}^n E[X_i \mid \Lambda] = \sum_{i=1}^n \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(V)+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Y_i}^2},$$

where U and $V = \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)$ are mutually independent uniform(0,1) random variables, Φ is the cdf of the $N(0,1)$ distribution and r_i is defined by

$$r_i = r\left(Y_i, \Lambda\right) = \frac{\text{cov}\left[Y_i, \Lambda\right]}{\sigma_{Y_i}\sigma_\Lambda} \geq 0.$$

A. Comonotonic upper bound

Since the cdfs F_{X_i} are strictly increasing and continuous, it follows from (3) and (9) that for $x \in \left(F_{\mathbb{S}^c}^{-1}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$, the cdf of the comonotonic sum $F_{\mathbb{S}^c}(x)$ can be found by solving

(12) $$\sum_{i=1}^n \alpha_i \, e^{E[Y_i]+\text{sign}(\alpha_i)\sigma_{Y_i}\Phi^{-1}(F_{\mathbb{S}^c}(x))} = x.$$

From Theorem 1 and (7), we find the following expression for the stop-loss premium at retention d with $F_{\mathbb{S}^c}^{-1+}(0) < d < F_{\mathbb{S}^c}^{-1}(1)$ for $\mathbb{S}^c$:

(13) $$E\left[\left(\mathbb{S}^c - d\right)_+\right] = \sum_{i=1}^n \alpha_i \, e^{E[Y_i]+\frac{\sigma_{Y_i}^2}{2}} \, \Phi\left[\text{sign}(\alpha_i)\sigma_{Y_i} - \Phi^{-1}(F_{\mathbb{S}^c}(d))\right] - d\left(1 - F_{\mathbb{S}^c}(d)\right).$$

B. Improved comonotonic upper bound

We now determine the cdf of $\mathbb{S}^u$ and the stop-loss premium $E\left[\left(\mathbb{S}^u - d\right)_+\right]$, where we condition on a normally distributed random variable Λ or equivalently on the uniform$(0,1)$ random variable introduced in Theorem 2:

(14) $$V = \Phi\left(\frac{\Lambda - E\left[\Lambda\right]}{\sigma_\Lambda}\right).$$

The conditional probability $F_{\mathbb{S}^u|V=v}(x)$ also denoted by $F_{\mathbb{S}^u}(x \mid V = v)$, is the cdf of a sum of n comonotonic random variables and follows for $F_{\mathbb{S}^u|V=v}^{-1}(0) < x < F_{\mathbb{S}^u|V=v}^{-1}(1)$, according to (4) and (10), implicitly from:

(15) $$\sum_{i=1}^n \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(v)+\text{sign}(\alpha_i)\sqrt{1-r_i^2}\,\sigma_{Y_i}\Phi^{-1}(F_{\mathbb{S}^u}(x|V=v))} = x.$$

The cdf of $\mathbb{S}^u$ is then given by

$$F_{\mathbb{S}^u}(x) = \int_0^1 F_{\mathbb{S}^u|V=v}(x)dv. \tag{16}$$

We now look for an expression for the stop-loss premium at retention d with $F^{-1}_{\mathbb{S}^u|V=v}(0) < d < F^{-1}_{\mathbb{S}^u|V=v}(1)$ for $\mathbb{S}^u$:

$$E\left[(\mathbb{S}^u - d)_+\right] = \int_0^1 E\left[(\mathbb{S}^u - d)_+ \mid V = v\right] dv = \sum_{i=1}^n \int_0^1 E\left[\left(F^{-1}_{X_i|\Lambda}(U \mid V = v) - d_i\right)_+\right] dv \tag{17}$$

with $d_i = F^{-1}_{X_i|\Lambda}\left(F_{\mathbb{S}^u}(d \mid V = v) \mid V = v\right)$ and with U a random variable which is uniformly distributed on $(0, 1)$. Since $\text{sign}(\alpha_i)F^{-1}_{X_i|\Lambda}(U \mid V = v)$ follows a lognormal distribution with mean and standard deviation:

$$\mu_v(i) = \ln|\alpha_i| + E\left[Y_i\right] + r_i\sigma_{Y_i}\Phi^{-1}(v), \quad \sigma_v(i) = \sqrt{1 - r_i^2}\,\sigma_{Y_i},$$

one obtains that

$$d_i = \alpha_i \exp\left[E\left[Y_i\right] + r_i\sigma_{Y_i}\Phi^{-1}(v) + \text{sign}(\alpha_i)\sqrt{1 - r_i^2}\,\sigma_{Y_i}\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(d)\right)\right]. \tag{18}$$

The well-known formula (7) then yields

$$E\left[(\mathbb{S}^u - d)_+ \mid V = v\right] = \sum_{i=1}^n \left[\text{sign}(\alpha_i)e^{\mu_v(i) + \frac{\sigma_v^2(i)}{2}}\Phi(\text{sign}(\alpha_i)d_{i,1}) - d_i\Phi(\text{sign}(\alpha_i)d_{i,2})\right],$$

with, according to (8),

$$d_{i,1} = \frac{\mu_v(i) + \sigma_v^2(i) - \ln|d_i|}{\sigma_v(i)}, \qquad d_{i,2} = d_{i,1} - \sigma_v(i).$$

Substitution of the corresponding expressions and integration over the interval $[0, 1]$ leads to the following result

$$\begin{aligned} E\left[(\mathbb{S}^u - d)_+\right] = &\sum_{i=1}^n \alpha_i e^{E[Y_i] + \frac{1}{2}\sigma_{Y_i}^2\left(1 - r_i^2\right)} \times \\ &\times \int_0^1 e^{r_i\sigma_{Y_i}\Phi^{-1}(v)}\Phi\left(\text{sign}(\alpha_i)\sqrt{1 - r_i^2}\,\sigma_{Y_i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(d)\right)\right) dv \\ &- d\left(1 - F_{\mathbb{S}^u}(d)\right). \end{aligned} \tag{19}$$

C. Lower bound

In this subsection, we take for simplicity of notation all $\alpha_i \geq 0$. Further, we assume that the conditioning variable Λ is normally distributed and has the right sign such that the correlation

coefficients r_i are all positive. These conditions ensure that $\mathbb{S}^\ell$ is the sum of n comonotonic random variables.
Since $E[X_i \mid \Lambda]$ is non-decreasing, we can obtain $F_{\mathbb{S}^\ell}(x)$ according to (5) and (11) from

$$\sum_{i=1}^{n} \alpha_i \, e^{E[Y_i]+r_i\sigma_{Y_i}\Phi^{-1}(F_{\mathbb{S}^\ell}(x))+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Y_i}^2} = x. \tag{20}$$

Moreover as $\mathbb{S}^\ell$ is the sum of n lognormally distributed random variables, the stop-loss premium at retention $d(>0)$ can be expressed explicitly by invoking Theorem 1 and (7):

$$E\left[\left(\mathbb{S}^\ell - d\right)_+\right] = \sum_{i=1}^{n} \alpha_i \, e^{E[Y_i]+\frac{1}{2}\sigma_{Y_i}^2} \Phi\left[r_i\sigma_{Y_i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(d)\right)\right] - d\left(1 - F_{\mathbb{S}^\ell}(d)\right). \tag{21}$$

IV Fixed strike Asian options in a Black & Scholes setting

In the Black & Scholes model, the price of a risky asset $\{S(t),\ t \geq 0\}$ under the risk-neutral measure Q follows a geometric Brownian motion process, with volatility σ and with drift equal to the risk-free force of interest r:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dB(t), \qquad t \geq 0,$$

where $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process under Q. Hence, the random variables $\frac{S(t)}{S(0)}$ are lognormally distributed with parameters $(r - \frac{\sigma^2}{2})t$ and $t\sigma^2$.

We now shall concentrate on bounds for the fixed strike Asian option by comonotonicity reasoning and by using the approach of Rogers & Shi which has been generalized by Nielsen and Sandmann (2002). We only write down the formulae of the forward starting Asian options.

A. Bounds based on comonotonicity reasoning

We remark that the Asian option pricing in the Black & Scholes setting is in fact a particular case of sums of lognormal variables in Section III. Indeed, let us look at the price of the Asian call option with exercise price K, maturity date T and averaging over n prices of the underlying with $T - n + 1 \geq 0$:

$$AC(n, K, T) = \frac{e^{-rT}}{n} E^Q\left[(\mathbb{S} - nK)_+\right]$$

with

$$\mathbb{S} = \sum_{i=0}^{n-1} S(T-i) = \sum_{i=0}^{n-1} S(0)e^{(r-\frac{\sigma^2}{2})(T-i)+\sigma B(T-i)}. \tag{22}$$

This can be rewritten as a sum of lognormal random variables: $\mathbb{S} = \sum_{i=0}^{n-1} \alpha_i e^{Y_i}$ with

$$\begin{aligned} Y_i &= \sigma B(T-i) \sim N(0, \sigma^2(T-i)) \\ \alpha_i &= S(0)e^{(r-\frac{\sigma^2}{2})(T-i)}. \end{aligned} \tag{23}$$

Lower bound

Lower bounds for $AC(n, K, T)$ can be obtained from Section III.C. We investigate different conditioning random variables Λ. Taking into account that we aim to derive a closed-form expression for the lower bound, we define Λ as a normal random variable given by

$$\Lambda = \sum_{i=0}^{n-1} \beta_i B(T-i), \qquad \beta_i \in \mathbb{R}^+. \tag{24}$$

The choice of the weights β_i is motivated by the reasoning that the quality of the stochastic lower bound $E^Q[\mathbb{S} \mid \Lambda]$ can be judged by its variance. To maximize the quality, this variance should be made as close as possible to var$[\mathbb{S}]$. In other words, the average value

$$E^Q\left[\mathrm{var}[\mathbb{S} \mid \Lambda]\right] = \mathrm{var}[\mathbb{S}] - \mathrm{var}\left[E^Q[\mathbb{S} \mid \Lambda]\right]$$

should be minimized.
Intuitively, to get the best lower bound, Λ and $\mathbb{S}$ should be as alike as possible. Therefore, we have selected the following two candidates for Λ which turn out to give very good results:

1. a linear transformation of a first order approximation to $\sum_{i=0}^{n-1} S(T-i)$ in (22), as proposed in a general setting by Kaas, Dhaene and Goovaerts (2000):

$$\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-i)} B(T-i), \tag{25}$$

2. the standardized logarithm of the geometric average $\mathbb{G} = \sqrt[n]{\prod_{i=0}^{n-1} S(T-i)}$ as in Nielsen and Sandmann (2002):

$$\Lambda = \frac{\ln \mathbb{G} - E^Q[\ln \mathbb{G}]}{\sqrt{\mathrm{var}[\ln \mathbb{G}]}} = \frac{1}{\sqrt{\mathrm{var}[\sum_{i=0}^{n-1} B(T-i)]}} \sum_{i=0}^{n-1} B(T-i), \tag{26}$$

where

$$\mathrm{var}[\sum_{i=0}^{n-1} B(T-i)] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \min(T-i, T-j) = n^2 T - \frac{n}{6}(n-1)(4n+1).$$

For general positive β_i, the variance of Λ is given by

$$\sigma_\Lambda^2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \beta_i \beta_j \min\left(T-i, T-j\right).$$

We have that $Y_i \mid \Lambda = \lambda$ is normally distributed with mean $r_i \frac{\sigma\sqrt{T-i}}{\sigma_\Lambda}\lambda$ and variance $\sigma^2(T-i)(1-r_i^2)$ where

$$r_i = \rho_{T-i} = \frac{\mathrm{cov}\left(B(T-i), \Lambda\right)}{\sqrt{T-i}\,\sigma_\Lambda} = \frac{\sum_{j=0}^{n-1} \beta_j \min\left(T-i, T-j\right)}{\sqrt{T-i}\,\sigma_\Lambda}. \tag{27}$$

Hence, for any random variable U which is uniformly distributed on the unit interval, we find from Theorem 2

$$\mathbb{S}^\ell \equiv \sum_{i=0}^{n-1} E^Q\left[S(T-i) \mid \Lambda\right] \stackrel{d}{=} S(0)\sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2}\rho_{T-i}^2)(T-i)+\sigma\rho_{T-i}\sqrt{T-i}\,\Phi^{-1}(U)}, \tag{28}$$

which is a sum of n comonotonic risks. Applying (21), we find the lower bound:

$$\begin{aligned} AC(n,K,T) &\geq \frac{e^{-rT}}{n}E^Q[(\mathbb{S}^\ell - nK)_+] \\ &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\,\Phi\left[\sigma\rho_{T-i}\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)\right] - e^{-rT}K\left(1-F_{\mathbb{S}^\ell}(nK)\right) \end{aligned} \tag{29}$$

which holds for any $K > 0$. In this case, $F_{\mathbb{S}^\ell}(nK)$ follows from (20) which now reads:

$$S(0)\sum_{i=0}^{n-1}\exp\left[\left(r-\frac{\sigma^2}{2}\rho_{T-i}^2\right)(T-i) + \sigma\,\rho_{T-i}\sqrt{T-i}\,\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)\right] = nK. \tag{30}$$

This lower bound differs for the two choices (25) and (26) of Λ, only by the expression (27) for the correlation coefficient ρ_{T-i}:

1. $\rho_{T-i} = \dfrac{\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-j)}\min\left(T-i,T-j\right)}{\sqrt{T-i}\,\sigma_\Lambda}$
   with
   $\sigma_\Lambda^2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(2T-i-j)}\min\left(T-i,T-j\right),$

2. $\rho_{T-i} = \dfrac{\sum_{j=0}^{n-1}\min\left(T-i,T-j\right)}{\sqrt{n^2T-\frac{n}{6}(n-1)(4n+1)}\sqrt{T-i}} = \dfrac{n\left(T-i\right)-\frac{(n-i-1)(n-i)}{2}}{\sqrt{n^2T-\frac{n}{6}(n-1)(4n+1)}\sqrt{T-i}}$
   since $\sigma_\Lambda = 1$.

We note that the closed-form solution of the lower bound in Nielsen and Sandmann (2002) is a special case of (29) and (30). We also noticed that the lower bound when conditioning on the geometric average coincides with the so-called "naive" approximation of Curran (1994). Moreover, choosing another normal conditioning variable Λ, formulae (29)-(30) still hold by substituting the right σ_Λ and ρ_{T-i}.

In fact, the lower bound can be written for a general normally distributed conditioning variable Λ, satisfying the assumptions of Theorem 2, as an average of Black & Scholes formulae for an artificial underlying asset of which the price process $\tilde{S}(t)$ is a geometric Brownian motion with $\tilde{S}(0) = S(0)$ and with a non-constant volatility $\tilde{\sigma}_i = \sigma\rho_{T-i}$ at time instance $T-i$:

$$\tilde{S}(T-i) = \tilde{S}(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i B(T-i)}.$$

The exercise prices under consideration are

$$\tilde{K}_i = F^{-1}_{E[S(T-i)|\Lambda]}\left(F_{\mathbb{S}^\ell(nK)}\right) = S(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i\sqrt{T-i}\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right)}.$$

Indeed, the lower bound $AC(n,K,T)$ can easily be transformed into

$$\frac{e^{-rT}}{n}\sum_{i=0}^{n-1}E^Q[(\tilde{S}(T-i)-\tilde{K}_i)_+]=\frac{1}{n}\sum_{i=0}^{n-1}\left(e^{-ri}\tilde{S}(0)\Phi\left(d_{1,i}\right)-e^{-rT}\tilde{K}_i\Phi\left(d_{2,i}\right)\right)$$

where

$$\begin{aligned} d_{1,i} &= \frac{\left(r+\frac{\tilde{\sigma}_i^2}{2}\right)(T-i)-\ln\left(\frac{\tilde{K}_i}{\tilde{S}(0)}\right)}{\tilde{\sigma}_i\sqrt{T-i}}=\tilde{\sigma}_i\sqrt{T-i}-\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right), \\ d_{2,i} &= d_{1,i}-\tilde{\sigma}_i\sqrt{T-i}=-\Phi^{-1}\left(F_{\mathbb{S}^\ell}(nK)\right). \end{aligned}$$

Comonotonic upper bound

We now rewrite the upper bound of Simon, Goovaerts and Dhaene (2000) for the price of an Asian call option in the present settings. From (13), we find

$$\begin{aligned} AC(n,K,T) &\leq \frac{e^{-rT}}{n}E^Q\left[(\mathbb{S}^c-nK)_+\right]=\frac{e^{-rT}}{n}E^Q\left[\left(\sum_{i=0}^{n-1}F^{-1}_{S(T-i)}(U)-nK\right)_+\right] \\ &= \frac{e^{-rT}}{n}\sum_{i=0}^{n-1}E^Q\left[\left(S(T-i)-F^{-1}_{S(T-i)}\left(F_{\mathbb{S}^c}(nK)\right)\right)_+\right] \\ &= \frac{S(0)}{n}\sum_{i=0}^{n-1}e^{-ri}\,\Phi\left[\sigma\sqrt{T-i}-\Phi^{-1}\left(F_{\mathbb{S}^c}(nK)\right)\right]-e^{-rT}K\left(1-F_{\mathbb{S}^c}(nK)\right), \end{aligned} \tag{31}$$

which holds for any $K>0$.

The remaining problem is how to calculate $F_{\mathbb{S}^c}(nK)$. The latter quantity follows from

$$\sum_{i=0}^{n-1}F^{-1}_{S(T-i)}\left(F_{\mathbb{S}^c}(nK)\right)=nK,$$

or, equivalently (see (12)),

$$S(0)\sum_{i=0}^{n-1}\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-i)+\sigma\sqrt{T-i}\,\Phi^{-1}\left(F_{\mathbb{S}^c}(nK)\right)\right]=nK.$$

In this way, Simon, Goovaerts and Dhaene (2000) found the smallest linear combination of prices of European call options that bounds the price of an European-style Asian option from above:

$$\frac{1}{n}\sum_{i=0}^{n-1}(S(0)e^{-ri}\Phi(d_{1i})-e^{-rT}K_i\Phi(d_{2i})),$$

with

$$K_i = F^{-1}_{S(T-i)}(F_{\mathbb{S}^c}(nK)) = S(0)e^{(r-\frac{\sigma^2}{2})(T-i)+\sigma\sqrt{T-i}\,\Phi^{-1}(F_{\mathbb{S}^c}(nK))}$$

$$d_{1i} = \frac{(r+\frac{\sigma^2}{2})(T-i) - \ln\left(\frac{K_i}{S(0)}\right)}{\sigma\sqrt{T-i}} = \sigma\sqrt{T-i} - \Phi^{-1}(F_{\mathbb{S}^c}(nK))$$

$$d_{2i} = d_{1i} - \sigma\sqrt{T-i} = -\Phi^{-1}(F_{\mathbb{S}^c}(nK)).$$

Using Lagrange optimization, Nielsen and Sandmann (2002) also obtained a similar expression for this optimal combination.

Improved comonotonic upper bound

As in Section III.B., we consider a normal conditioning random variable. An improved comonotonic upper bound for the Asian option price is

$$AC(n,K,T) = \frac{e^{-rT}}{n}E^Q\left[(\mathbb{S}-nK)_+\right] \leq \frac{e^{-rT}}{n}E^Q\left[(\mathbb{S}^u-nK)_+\right], \tag{32}$$

where according to (19) and (14):

$$\begin{aligned} &E^Q\left[(\mathbb{S}^u-nK)_+\right] \\ &= \sum_{i=0}^{n-1} S(0)e^{r(T-i)}e^{-\frac{\sigma^2}{2}\rho^2_{T-i}(T-i)} \\ &\quad \times \int_0^1 e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho^2_{T-i}}\,\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right)dv \\ &\quad -nK\left(1-F_{\mathbb{S}^u}(nK)\right). \end{aligned} \tag{33}$$

The conditional distribution $F_{\mathbb{S}^u|V=v}(nK)$ follows from (18):

$$nK = \sum_{i=0}^{n-1}\alpha_i \exp\left[\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v) + \sqrt{1-\rho^2_{T-i}}\,\sigma\sqrt{T-i}\,\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right] \tag{34}$$

where $\alpha_i = S(0)e^{(r-\frac{\sigma^2}{2})(T-i)}$, (23), and the cdf of $\mathbb{S}^u$ is obtained from (16).

We found that the conditioning variable

$$\Lambda = \sum_{k=1}^{T}\beta_k W_k, \quad \text{with } W_k \text{ i.i.d. } N(0,1) \text{ such that } B(T-i) \overset{d}{=} \sum_{k=1}^{T-i} W_k,\ i=0,...n-1, \tag{35}$$

with all β_k equal to a same constant (for simplicity taken equal to one) leads to a sharper upper bound than other choices for β_k or than the conditioning variables in the lower bound.
For $\Lambda = \sum_{k=1}^{T} W_k \overset{d}{=} B(T)$ the correlation terms have the form:

$$r_i = \rho_{T-i} = \frac{\text{cov}\left(B(T-i),\Lambda\right)}{\sqrt{T-i}\,\sigma_\Lambda} = \frac{T-i}{\sqrt{T-i}\sqrt{T}} = \frac{\sqrt{T-i}}{\sqrt{T}},\ i=0,\ldots,n-1, \tag{36}$$

and the dependence structure of the terms in the sum $\mathbb{S}^u$ corresponds better to that of the terms in the sum $\mathbb{S}$ than for other choices of Λ. Investigating the correlations

$$r\left[F^{-1}_{S(T-i)|\Lambda}(U), F^{-1}_{S(T-j)|\Lambda}(U)\right] = \frac{e^{[\rho_{T-i}\rho_{T-j}+\sqrt{1-\rho^2_{T-i}}\sqrt{1-\rho^2_{T-j}}]\sigma^2\sqrt{T-i}\sqrt{T-j}}-1}{\sqrt{e^{\sigma^2(T-i)}-1}\sqrt{e^{\sigma^2(T-j)}-1}}$$

$$r\left[S(T-i), S(T-j)\right] = \frac{e^{\sigma^2\min(T-i,T-j)}-1}{\sqrt{e^{\sigma^2(T-i)}-1}\sqrt{e^{\sigma^2(T-j)}-1}},$$

it can be seen that for ρ_{T-i} given by (36) these correlations not only coincide for $i=j$ but also when one of the indices i or j equals zero. Moreover, for $i \neq j$, the differences

$$\left|[\rho_{T-i}\rho_{T-j}+\sqrt{1-\rho^2_{T-i}}\sqrt{1-\rho^2_{T-j}}]\sigma^2\sqrt{T-i}\sqrt{T-j}-\sigma^2\min(T-i,T-j)\right|$$

are small for all i and j in $\{0,\ldots,n-1\}$ in comparison to other choices of Λ.

As in case of the lower bound, we can rewrite the upper bound as an expression of Black & Scholes formulae for an underlying asset $\tilde{S}(t)$ with $\tilde{S}(0) = S(0)$ and with volatilities $\tilde{\sigma}_i = \sigma\sqrt{1-\rho^2_{T-i}}$:

$$\tilde{S}(T-i) = \tilde{S}(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i B(T-i)}.$$

Indeed, an equivalent expression for (33) is rewritten as

$$\begin{aligned}\frac{e^{-rT}}{n}E^Q\left[(\mathbb{S}^u-nK)_+\right] &= \int_0^1 \frac{1}{n}\sum_{i=0}^{n-1} e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)-\frac{\sigma^2}{2}\rho^2_{T-i}(T-i)} \\ &\quad \times\left\{e^{-ri}\tilde{S}(0)\Phi\left(d_{1,i}(v)\right)-e^{-rT}\tilde{K}_i(v)\Phi\left(d_{2,i}(v)\right)\right\}dv\end{aligned}$$

with exercise prices defined by

$$\tilde{K}_i(v) = S(0)e^{(r-\frac{\tilde{\sigma}_i^2}{2})(T-i)+\tilde{\sigma}_i\sqrt{T-i}\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)}$$

and

$$\begin{aligned} d_{1,i}(v) &= \frac{\left(r+\frac{\tilde{\sigma}_i^2}{2}\right)(T-i)-\ln\left(\frac{\tilde{K}_i(v)}{\tilde{S}(0)}\right)}{\tilde{\sigma}_i\sqrt{T-i}} = \tilde{\sigma}_i\sqrt{T-i}-\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right) \\ d_{2,i}(v) &= d_{1,i}(v)-\tilde{\sigma}_i\sqrt{T-i} = -\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right). \end{aligned}$$

B. Bounds based on the Rogers & Shi approach

Following the ideas of Rogers and Shi (1995), we derive an upper bound based on the lower bound. Indeed, applying the following general inequality for any random variable Y and Z from Rogers and Shi (1995):

$$0 \leq E\left[E\left[Y^+ \mid Z\right]-E\left[Y \mid Z\right]^+\right] \leq \frac{1}{2}E\left[\sqrt{\text{var}(Y \mid Z)}\right]$$

to the case of Y being $\sum_{i=0}^{n-1} S(T-i) - nK$ and Z being our conditioning variable Λ given by (24), we obtain an error bound

$$0 \leq E^Q \left[E^Q \left[(\mathbb{S} - nK)^+ \mid \Lambda \right] - (\mathbb{S}^\ell - nK)^+ \right] \leq \frac{1}{2} E^Q \left[\sqrt{\text{var}(\mathbb{S} \mid \Lambda)} \right]. \tag{37}$$

Consequently, we find as upper bound for the arithmetic Asian option

$$AC(n,K,T) \leq \frac{e^{-rT}}{n} \left\{ E^Q \left[(\mathbb{S}^\ell - nK)^+ \right] + \frac{1}{2} E^Q \left[\sqrt{\text{var}(\mathbb{S} \mid \Lambda)} \right] \right\}. \tag{38}$$

Using properties of lognormal distributed variables, the second term on the right hand side can be written out explicitly, giving some lenghty, analytical, computable expression:

$$\begin{aligned} E^Q \left[\sqrt{\text{var}(\mathbb{S} \mid \Lambda)} \right] &= E^Q \left[\left(E^Q \left[\mathbb{S}^2 \mid \Lambda \right] - E^Q \left[\mathbb{S} \mid \Lambda \right]^2 \right)^{1/2} \right] \\ &= E^Q \left[\left(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E^Q \left[S(T-i) S(T-j) \mid \Lambda \right] - \left(\mathbb{S}^\ell \right)^2 \right)^{1/2} \right], \end{aligned} \tag{39}$$

where the first term in the expectation in the right hand side equals

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \alpha_i \alpha_j \exp \left(r_{ij} \sigma \sigma_{ij} \Phi^{-1}(U) + \frac{1}{2}(1 - r_{ij}^2) \sigma^2 \sigma_{ij}^2 \right), \tag{40}$$

with

$$\alpha_i \alpha_j = S(0)^2 \exp \left[(r - \frac{\sigma^2}{2})(2T - i - j) \right], \tag{41}$$

$$\sigma_{ij} = \sqrt{(T-i) + (T-j) + 2 \min(T-i, T-j)}, \tag{42}$$

$$r_{ij} = \frac{\sqrt{T-i}}{\sigma_{ij}} \rho_{T-i} + \frac{\sqrt{T-j}}{\sigma_{ij}} \rho_{T-j}, \tag{43}$$

and where U is uniformly distributed on the interval $(0,1)$.

This upper bound also holds when starting from a lower bound with a normal conditioning variable Λ different from (24). This allows us to take the minimum over several upper bounds. Note also that the error bound (37) is independent of the strike K.

For Λ given by (26), Nielsen and Sandmann (2002) were able to strengthen the error bound of Rogers and Shi. We show that also for Λ given by (25) this technique works to strengthen the error bound (37) and hence to sharpen the upper bound (38).
Using the property that $e^x \geq 1 + x$ and relations (22)-(23) and (25), we obtain

$$\mathbb{S} = \sum_{i=0}^{n-1} \alpha_i e^{Y_i} \geq \sum_{i=0}^{n-1} \alpha_i + S(0)\sigma \underbrace{\sum_{i=0}^{n-1} e^{(r - \frac{\sigma^2}{2})(T-i)} B(T-i)}_{\Lambda}.$$

Hence $\mathbb{S} \geq nK$ when Λ is larger than $\frac{nK - \sum_{i=0}^{n-1} \alpha_i}{S(0)\sigma}$. Thus in case of Λ being a linear transformation of the first order approximation (FA) of $\mathbb{S}$, we have for $\Lambda \geq d_{FA}$ with

$$d_{FA} = \frac{nK - \sum_{i=0}^{n-1} S(0) e^{(r - \frac{\sigma^2}{2})(T-i)}}{S(0)\sigma}, \tag{44}$$

that

$$E^Q[(\mathbb{S} - nK)_+ \mid \Lambda] = E^Q[\mathbb{S} - nK \mid \Lambda] = \left(\mathbb{S}^\ell - nK\right)_+ .$$

In general, for $d \in \mathbb{R}$ such that $\Lambda \geq d$ implies that $\mathbb{S} \geq nK$, it follows that by using the notation $f_\Lambda(\cdot)$ for the normal density function of Λ:

$$\begin{aligned}
0 &\leq E^Q\left[E^Q[(\mathbb{S} - nK)_+ \mid \Lambda] - \left(\mathbb{S}^\ell - nK\right)_+\right] \\
&= \int_{-\infty}^{d} \left(E^Q[(\mathbb{S} - nK)_+ \mid \Lambda = \lambda] - \left(E^Q\left[\mathbb{S} \mid \Lambda = \lambda\right] - nK\right)_+\right) f_\Lambda(\lambda) d\lambda \\
&\leq \frac{1}{2} \int_{-\infty}^{d} \left(\mathrm{var}\left(\mathbb{S} \mid \Lambda = \lambda\right)\right)^{\frac{1}{2}} f_\Lambda(\lambda) d\lambda \\
&\leq \frac{1}{2} \left(E^Q\left[\mathrm{var}\left(\mathbb{S} \mid \Lambda\right) 1_{\{\Lambda < d\}}\right]\right)^{\frac{1}{2}} \left(E^Q\left[1_{\{\Lambda < d\}}\right]\right)^{\frac{1}{2}} , \qquad (45)
\end{aligned}$$

where Hölder's inequality has been applied in the last inequality and where $1_{\{\Lambda < d\}}$ is the indicator function.

The upper bound (38) corresponds to the limiting case where d equals infinity. Further note that in contrast to (37) the error bound now depends on K through d.

We stress that the error bound (45) holds for any conditioning random normal variable Λ that satisfies the assumptions of Theorem 2 and for which there exists an integration bound d such that $\Lambda \geq d$ implies $\mathbb{S} \geq nK$. For Λ given by (26), Nielsen and Sandmann found that the corresponding d is given by

$$d = \frac{n \ln\left(\frac{K}{S(0)}\right) - \sum_{i=0}^{n-1} \left(r - \frac{\sigma^2}{2}\right)(T - i)}{\sigma\sqrt{n^2 T - \frac{1}{6} n(n-1)(4n+1)}}, \tag{46}$$

which we denote by d_{GA} for reminding the fact that Λ is the standardized logarithm of the geometric average.

Now we shall derive an easily computable expression for (45). The second expectation term in the product (45) equals $F_\Lambda(d)$ where $F_\Lambda(\cdot)$ denotes the normal cumulative distribution function of Λ. The first expectation term in the product (45) can be expressed as

$$E^Q\left[\mathrm{var}\left(\mathbb{S} | \Lambda\right) 1_{\{\Lambda < d\}}\right] = E^Q\left[E^Q[\mathbb{S}^2 | \Lambda] 1_{\{\Lambda < d\}}\right] - E^Q\left[(E^Q[\mathbb{S} | \Lambda])^2 1_{\{\Lambda < d\}}\right]. \tag{47}$$

The second term of the right-hand side of (47) can according to (28) be rewritten as

$$\begin{aligned}
E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d\}}\right] &= \int_{-\infty}^{d} (E^Q[\mathbb{S}|\Lambda = \lambda])^2 f_\Lambda(\lambda) d\lambda \\
&= S(0)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{r(2T-i-j) - \frac{\sigma^2}{2}\left(\rho_{T-i}^2 (T-i) + \rho_{T-j}^2 (T-j)\right)} \int_{-\infty}^{d} e^{\sigma\left(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}\right)\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda,
\end{aligned}$$

(48)

where we recall that $\Phi^{-1}(v) = \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. Applying the equality

$$\int_{-\infty}^{d} e^{b\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda = e^{\frac{b^2}{2}} \Phi\left(d^* - b\right), \quad d^* = \frac{d - E^Q[\Lambda]}{\sigma_\Lambda}, \tag{49}$$

with $b = \sigma\left(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}\right)$ we can express $E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d\}}\right]$ as

$$S(0)^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}} \Phi\left(d^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j})\right). \tag{50}$$

To transform the first term of the right-hand side of (47) we invoke (40)-(43) and apply (49) with $b = r_{ij}\sigma\sigma_{ij} = \sigma\left(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}\right)$:

$$\begin{aligned}
& E^Q\left[E^Q[\mathbb{S}^2 \mid \Lambda] 1_{\{\Lambda<d\}}\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \int_{-\infty}^{d} E^Q\left[S(T-i)S(T-j)|\Lambda = \lambda\right] f_\Lambda(\lambda) d\lambda \\
&= S(0)^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(2T-i-j)+\frac{1}{2}\left(1-r_{ij}^2\right)\sigma^2\sigma_{ij}^2} \int_{-\infty}^{d} e^{r_{ij}\sigma\sigma_{ij}\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda \\
&= S(0)^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\min(T-i,T-j)} \Phi\left(d^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j})\right). \qquad (51)
\end{aligned}$$

Combining (50) and (51) into (47), and then substituting $F_\Lambda(d)$ and (47) into (45) we get the following expression for the error bound, shortly denoted by $\varepsilon(d)$

$$\begin{aligned}
\varepsilon(d) = & \frac{S(0)}{2} \left\{F_\Lambda(d)\right\}^{\frac{1}{2}} \times \\
& \times \left\{ \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{r(2T-i-j)+\sigma^2\rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}} \Phi\left(d^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j})\right) \times \right. \\
& \left. \times \left(e^{\sigma^2\left(\min(T-i,T-j) - \rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}\right)} - 1\right) \right\}^{\frac{1}{2}}.
\end{aligned}$$

This error bound coincides with the one found in Nielsen and Sandmann (2002) for the special choice (26) for Λ and the corresponding d_{GA} (46).

C. Partially exact/comonotonic upper bound

We combine the technique for obtaining an improved comonotonic upper bound by conditioning on some normally distributed random variable Λ and the idea of Nielsen and Sandmann (2002) described in the previous subsection, in order to develop another upper bound. This so-called partially exact/comonotonic upper bound consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part. This idea of decomposing the calculations in an exact part and an approximating part goes at least back to Curran (1994). This upper bound also corresponds to the upper bound denoted by $C_A^{u,G}$ in the paper of Nielsen and Sandmann (2002).

For any normally distributed random variable Λ, with cdf $F_\Lambda(\cdot)$, for which there exists a d such that $\Lambda \geq d$ implies $\mathbb{S} \geq nK$ and which satisfies the assumptions of Theorem 2, the second term in the equality

$$\begin{aligned}
&\frac{e^{-rT}}{n} E^Q[(\mathbb{S}-nK)_+] = \frac{e^{-rT}}{n} E^Q[E^Q[(\mathbb{S}-nK)_+ \mid \Lambda]] \\
(52)\quad &= \frac{e^{-rT}}{n} \left\{ \int_{-\infty}^{d} E^Q[(\mathbb{S}-nK)_+ \mid \Lambda=\lambda] dF_\Lambda(\lambda) + \int_{d}^{+\infty} E^Q[\mathbb{S}-nK \mid \Lambda=\lambda] dF_\Lambda(\lambda) \right\}
\end{aligned}$$

can be written in closed-form along similar lines as (48)-(50):

$$\begin{aligned}
&\frac{e^{-rT}}{n} \int_{d}^{+\infty} E^Q[\mathbb{S} \mid \Lambda=\lambda] f_\Lambda(\lambda) d\lambda - e^{-rT} K(1-F_\Lambda(d)) \\
&= \frac{e^{-rT}}{n} \sum_{i=0}^{n-1} S(0) e^{(r-\frac{1}{2}\sigma^2\rho_{T-i}^2)(T-i)} \int_{d}^{+\infty} e^{\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)} f_\Lambda(\lambda) d\lambda - e^{-rT} K(1-\Phi(d^*)) \\
(53)\quad &= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-ri} \Phi(\rho_{T-i}\sigma\sqrt{T-i} - d^*) - e^{-rT} K \Phi(-d^*),
\end{aligned}$$

where $d^* = \dfrac{d - E^Q[\Lambda]}{\sigma_\Lambda}$ and $v = \dfrac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$.
In the first term of (52) we replace $\mathbb{S}$ by $\mathbb{S}^u$ in order to obtain an upper bound and apply (33) but now with an integral from zero to $\Phi(d^*)$:

$$\begin{aligned}
&\frac{e^{-rT}}{n} \int_{-\infty}^{d} E^Q[(\mathbb{S}-nK)_+ \mid \Lambda=\lambda] f_\Lambda(\lambda) d\lambda \\
&\leq \frac{e^{-rT}}{n} \int_{-\infty}^{d} E^Q[(\mathbb{S}^u-nK)_+ \mid \Lambda=\lambda] f_\Lambda(\lambda) d\lambda = \frac{e^{-rT}}{n} \int_0^{\Phi(d^*)} E^Q[(\mathbb{S}^u-nK)_+ \mid V=v]\, dv \\
&= \frac{S(0)}{n} \sum_{i=0}^{n-1} e^{-ri} e^{-\frac{\sigma^2}{2}\rho_{T-i}^2(T-i)} \\
&\qquad \times \int_0^{\Phi(d^*)} e^{\rho_{T-i}\,\sigma\sqrt{T-i}\,\Phi^{-1}(v)} \Phi\left(\sqrt{1-\rho_{T-i}^2}\,\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(nK)\right)\right) dv \\
&\quad - e^{-rT} K \left(\Phi(d^*) - \int_0^{\Phi(d^*)} F_{\mathbb{S}^u|V=v}(nK)\, dv\right).
\end{aligned}$$

(54)

For the random variables Λ given by (25) and (26) we derived a d, see (44) and (46), and thus we can compute the new upper bound. Recall that these choices of Λ do not lead to the best improved comonotonic upper bound. The best choice is $\Lambda = B(T)$ for which we do not find the necessary

d in this new upper bound. However, we expect that the contribution of the exact part (53) which is the second term in (52) will compensate for the somewhat lower quality of the $\mathbb{S}^u$.
Finally, we note that the bound $C_A^{u,G}$ in Nielsen and Sandmann (2002) was derived for the special conditioning variable Λ given by (26), and that they need an optimization algorithm to find the weights a_i such that their upper bound for the first term in (52), namely

$$\frac{e^{-rT}}{n}\sum_{i=0}^{n-1}\int_{-\infty}^{d} E^Q[(S(T-i)-a_i nK)_+ \mid \Lambda=\lambda] f_\Lambda(\lambda)d\lambda,$$

is minimized. With our method we explicitly have the optimal weights a_i for a given λ or v:

$$\begin{aligned} a_i &= \frac{1}{nK} F^{-1}_{S(T-i)|\Lambda=\lambda}(F_{\mathbb{S}^u|V=v}(nK)) \\ &= \frac{S(0)}{nK} e^{(r-\frac{\sigma^2}{2})(T-i)+\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)+\sqrt{1-\rho_{T-i}^2}\,\sigma\sqrt{T-i}\,\Phi^{-1}(F_{\mathbb{S}^u|V=v}(nK))}. \end{aligned}$$

D. General remarks

In this section we summarize some general remarks:

1. Denoting the price of an arithmetic European-style Asian *put* option with exercise date T, n averaging dates and fixed exercise price K by $AP(n,K,T)$, we find from the put-call parity at the present:

   (55) $$AC(n,K,T) - AP(n,K,T) = \frac{S(0)}{n}\frac{1-e^{-rn}}{1-e^{-r}} - e^{-rT}K.$$

   Hence, we can derive bounds for the Asian put option from the bounds for the call. These bounds for the put option coincide with the bounds that are obtained by applying the theory of comonotonic bounds or the Rogers and Shi approach directly to Asian put options. This stems from the fact that the put-call parity also holds for these bounds.
   Note that for numerical computations, if n and T are expressed in days then r should be interpreted as a daily compounded interest rate which equals a yearly compounded interest rate divided by the number of (trading) days per year.

2. The case of a continuous dividend yield δ can easily be dealt with by replacing the interest rate r by $r-\delta$.

3. When the number of averaging dates n equals 1, the Asian call option reduces to a European call option. It can be proven that in this case the upper and the lower bounds for the price of the Asian option both reduce to the Black & Scholes formula for the price of a European call option. For the improved comonotonic upper bound this is true thanks to the special choice of $\Lambda = B(T)$, while for the upper bound (38) it holds since the conditioning variable Λ equals $\beta_T B(T)$.

4. The lower and upper bounds are derived for forward starting Asian options but they can easily be adapted to hold for Asian options in progress. In this case $T-n+1 \leq 0$ and only the prices of $S(1), \ldots, S(T)$ remain random such that the price of the option reads:

$$\begin{aligned} AC(n,K,T) &= \frac{e^{-rT}}{n} E^Q \left[\left(\sum_{i=0}^{n-1} S(T-i) - nK \right)_+ \right] \\ &= \frac{e^{-rT}}{n} E^Q \left[\left(\sum_{i=0}^{T-1} S(T-i) - \left(nK - \sum_{i=T}^{n-1} S(T-i) \right) \right)_+ \right]. \end{aligned}$$

Thus substituting $nK - \sum_{i=T}^{n-1} S(T-i)$ for nK and summing for the average over i from zero to $T-1$ instead of $n-1$ the desired bounds follow.

E. Numerical illustration

In this section we give a number of numerical examples in the Black & Scholes setting. We discuss our results and compare them to those found in the literature and to the Monte Carlo price. Further, we approximate $\mathbb{S}$ by a lognormal distribution which is the closest in the Kullback-Leibler sense. We also measure the closeness of the lower and upper bounds in the distributional sense.

Comparing bounds

In this section we discuss our results and compare them with those of Jacques (1996) where the distribution of the sum $\mathbb{S}$ of lognormals, (22), entering in the arithmetic Asian option was approximated by means of the lognormal (LN) and the inverse Gaussian (IG) distribution. For the comparison we also included the upper bounds based on the method of Rogers and Shi, (38), and of Nielsen and Sandmann, (45).
We show here one set of numerical experiments where we consider a forward starting Asian option with fixed strike having the same data as in the paper of Jacques (1996): an initial stock price $S(0) = 100$, an annual (nominal, daily compounded) interest rate of 9$\%$ (i.e. $r = \ln\left(1 + \frac{0.09}{365}\right)$ daily), a maturity of 120 days and an averaging period n of 30 days. The values of the volatility σ are on annual basis. As a benchmark we included the price obtained via Monte Carlo simulation by adapting the control variate technique of Kemna and Vorst (1990) to discretely sampled Asian options. The number of simulated Monte Carlo paths was $10\,000$.

As we see from Table 1, the lower bounds LBFA and LBGA, for the conditioning on Λ given by (25) and (26), are equal up to five decimals. They both perform much better than the lower bound LBB_T where we conditioned on $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$ (cfr. (35)), as was expected.
The improved comonotonic upper bound ICUB, (32)-(33), is smaller than the comonotonic upper bound CUB, (31), from Simon et al. (2000), as stated in the theory. The Rogers and Shi upper bound UBFA, (38), performs better than ICUB, except when the option is deep in-the-money. We also included the upper bound UBB_T based on the lower bound LBB_T according to (38) but with conditioning on $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$ (cfr. (35)). The bad performance is due to the fact that $B(T)$ differs much from $\mathbb{S}$ for n larger than one and hence $E^Q\left[\text{var}^{1/2}(\mathbb{S} \mid B(T))\right]$ is large, while

for Λ, (25) or (26), this term $E^Q\left[\text{var}^{1/2}(\mathbb{S}\,|\,\Lambda)\right]$ is very small because Λ en $\mathbb{S}$ are very much alike. It seems that the relative difference between a lower bound and an upper bound increases with K. For the upper bounds UBFA and UBB_T this is clear, since for different values of K a same constant is added while the value of the lower bound is decreasing.
The improved upper bound UBGA_d which is based on the lower bound LBGA plus a pricing error cfr. (45)-(46), performs the best of all upper bounds considered. However, UBFA_d which is the lower bound LBFA plus the error bound $\varepsilon(d_{FA})$ given by (44) and (52), performs good as well. For this set of parameters, the values for the partially exact/comonotonic upper bound PECUB, (52)-(54), are smaller than those for the improved comonotonic upper bound ICUB but, as the results in Table 1 show for the case of Λ given by (26), they are not that good as we would have expected.
Comparing UBFA with UBFA_d, we note that making the error bound dependent on the exercise price K has led to an improvement except for a volatility equal to 0.4 and $K = 80$. An explanation is that the Hölder inequality introduces an additional error which can be larger than the improvement that is obtained by introducing the integration bound d.

Table 1 also reveals that the lognormal (LN) approximation as well as the inverse Gaussian (IG) approximation of Jacques (1996) underestimate systematically the price of the option since the prices are smaller than the (comonotonic) lower bounds LBFA and LBGA. Moreover, they are lower than the respective Monte Carlo prices. Further, note that the precision of the simulated prices decreases as the volatility σ increases. The Monte Carlo approach systematically seems to underestimate the true price, especially for at- and out-of-the-money options for which the Monte Carlo price falls slightly below the lower bounds.

The effect of the averaging period and of interest rates on the bounds

For different sets of parameters, we have computed the lower and the upper bounds together with the price obtained by Monte Carlo simulation[1]. The latter is based on generating $10\,000$ paths. This has been done in particular for four different options: the first with expiration date at time $T = 120$ and 30 averaging days, the second with expiration at time $T = 60$ and 30 averaging days, the third one with again expiration time $T = 120$ but only 10 averaging days, and as the last one we considered the case where averaging was done over the whole period of 120 days. In all cases we considered the 4 following exercise prices K: 80, 90, 100 and 110, three values (0.2, 0.3 and 0.4) for the volatility σ, and the two different flat risk-free interest rates r: 5% and 9% yearly. The initial stock price was fixed at $S(0) = 100$.

The absolute and relative differences between the best upper and lower bound increase with the volatility and with the exercise price, but decrease with the interest rate. The results further suggest that all intervals are sharper for options that are in-the-money. For fixed maturity, the length of the intervals reduces with the number of averaging dates. However for a fixed averaging period the effect of the maturity date seems to be less clear.

We can conclude that the difference between the lower bounds LBGA and LBFA is overall practically zero. The upper bound UBGA_d is in general the best but for example when $r = 0.05$, $K = 100$ and $\sigma = 0.4$, UBFA_d turns out to be smaller than UBGA_d.

[1]The tables with the results discussed in this paragraph are available on request.

Comparison of lower and upper bounds as in Nielsen and Sandmann (2002) with our bounds

In this section we use the data from Nielsen and Sandmann (2002) in order to compare their different upper bounds with our results (we kept their notation for respective bounds). They give as input data: $\sigma = 0.25$, $r = 0.04$, $S(0) = 100$, $T = 3$ years. Note that they use price averaging over the whole period ($n = 3$ years) where averaging takes place each month (in the previous sections the averaging was done daily).
The first column of Table 2 shows the selection of strike prices from Nielsen and Sandmann (2002). In addition to the strike prices used in the above sections we also included $K = 50$ and $K = 200$ as examples of extreme in- and out-of-the-money options. The column RS contains the Rogers and Shi upper bound based on the geometric average conditioning variable and with d equal to infinity.
Nielsen and Sandmann (2002) derive another upper bound $C_A^{u,G}$ which depends on coefficients a_i satisfying $\sum_{i=1}^n a_i = 1$. The last three columns in Table 2 show the bounds $C_A^{u,G}$ for different choices of coefficients a_i. The columns labelled as $C_A^{*,G}$ and $C_A^{N,G}$ are computed for the choice of $a_i = a_i^*$ (special choice by Nielsen and Sandmann) and $a_i = \frac{1}{n}$, respectively. The column $C_A^{**,G}$ presents the results for the optimal sequence of the weights a_i in relation to the $C_A^{u,G}$ bound (i.e. the sequence which minimizes the upper bound $C_A^{u,G}$).
We note again that the partially exact/comonotonic upper bound PECUB is smaller and thus better than the improved comonotonic upper bound ICUB for exercise prices in the range 50–150 (not all values are reported in Table 2), but for deeply out-of-the-money options there is a switch and ICUB becomes better and even for $K = 200$ outperforms all other the upper bounds including the choices of Nielsen and Sandmann.

Distributional distance between the bounds and lognormal approximation of $\mathbb{S}$

As already mentioned, the sum of lognormal random variables is not lognormally distributed. However, in practice it is often claimed to be approximately lognormal. In this section we aim to quantify the distance between the distribution of $\mathbb{S}$, (6), and the lognormal family of distributions by means of the so-called Kullback-Leibler information. We also use the Hellinger distance in order to measure the closeness of the derived lower and upper bounds. This section uses the ideas from Brigo and Liinev (2002) and we refer to this paper for definitions.

Firstly, note that it is possible to calculate the Kullback-Leibler distance (KLI) of the distribution of the sum $\mathbb{S}$ from the lognormal family of distributions $\mathcal{L}$ in the following way

$$\begin{aligned} D(p(x), \mathcal{L}) &= E_p[\ln p] + \frac{1}{2} + E_p\left[\ln\left(\frac{x}{S(0)}\right)\right] \\ &+\frac{1}{2}\ln\left(2\pi S(0)^2\left[E_p\left[\ln^2\left(\frac{x}{S(0)}\right)\right] - \left(E_p\left[\ln\left(\frac{x}{S(0)}\right)\right]\right)^2\right]\right) , \end{aligned} \tag{56}$$

where $p(x)$ denotes the density function of $\mathbb{S}$, and E_p is the expectation with respect to $\mathbb{S}$. This distance is readily computed, once one has an estimate of the true $\mathbb{S}$ density and of its first two log-moments.

$T = 120$, $n = 30$, $r = \ln(1 + 0.09/365)$ daily, $S(0) = 100$

σ	K	LN	IG	MC (S.E.(10^{-4}))	LBB_T	LBFA	LBGA	UBGA_d	UBFA_d	UBFA	PECUB	ICUB	CUB	UBB_T
0.2	80	–	–	22.00271 (2.5)	21.994822	22.002619	22.002619	22.002732	22.002849	22.014767	22.004625	22.006032	22.008177	23.446236
	90	12.68	12.68	12.76012 (2.6)	12.691751	12.760052	12.760053	12.761283	12.761506	12.772219	12.778069	12.786728	12.803051	14.143164
	100	5.46	5.46	5.521652 (2.5)	5.364993	5.521689	5.521689	5.526257	5.526389	5.533856	5.566340	5.580651	5.616195	6.816407
	110	1.63	1.63	1.652697 (2.0)	1.518289	1.652807	1.652806	1.661491	1.661639	1.664974	1.695799	1.704168	1.735318	2.969703
0.3	80	–	–	22.30976 (5.8)	22.250172	22.309736	22.309736	22.311225	22.311808	22.337168	22.325349	22.333495	22.348143	24.428128
	90	13.85	13.85	13.92461 (5.9)	13.763614	13.924578	13.924579	13.929696	13.930099	13.952005	13.968496	13.985921	14.023081	15.941570
	100	7.48	7.49	7.534506 (5.8)	7.295732	7.534676	7.534676	7.545641	7.545771	7.562103	7.603959	7.624473	7.678566	9.473688
	110	3.48	3.49	3.517352 (5.1)	3.288965	3.517536	3.517535	3.534765	3.535066	3.544963	3.589000	3.604201	3.656598	5.466921
0.4	80	–	–	23.03488 (10.7)	22.894509	23.034765	23.034765	23.039974	23.085083	23.083564	23.072463	23.088993	23.122019	25.800008
	90	15.36	15.37	15.42367 (10.8)	15.172741	15.423789	15.423789	15.435454	15.435878	15.472586	15.493971	15.518613	15.575829	18.078240
	100	9.51	9.53	9.563843 (10.5)	9.244120	9.564114	9.564114	9.584043	9.584080	9.612911	9.658116	9.684280	9.756619	12.149619
	110	5.48	5.49	5.517215 (9.7)	5.199653	5.517573	5.517573	5.545909	5.546323	5.566370	5.616391	5.637784	5.710355	8.105152

Table 1: Bounds versus LN and IG approximations of Jacques (1996).

$T = 3$ years, $n = 3$ years, $r = 0.04$ yearly, $S(0) = 100$

σ	K	LBFA	$C_A^{l,Z}$	UBFA_d	UBFA	PECUB	ICUB	CUB	RS	$C_A^{u,Z}$	$C_A^{**,G}$	$C_A^{*,G}$	$C_A^{N,G}$
0.25	50	50.04725	50.0472	50.05985	50.55569	50.05167	50.05653	50.06584	50.6536	50.0488	50.0518	50.0535	50.0641
	80	24.74574	24.7471	24.83418	25.25418	25.02989	25.21253	25.50575	25.3535	24.8222	25.0424	25.0931	25.2908
	90	17.93115	17.9343	18.06319	18.43958	18.40466	18.63671	19.06655	18.5406	18.0582	18.4309	18.495	18.6188
	100	12.47590	12.4743	12.65653	12.98433	13.11488	13.33504	13.85613	13.0807	12.649	13.1516	13.2158	13.2088
	110	8.38599	8.383	8.62056	8.89442	9.12588	9.28428	9.83599	8.9894	8.611	9.1717	9.2261	9.1827
	200	0.11830	0.1159	0.61035	0.62674	0.25144	0.20810	0.28556	0.7223	0.6962	0.2662	0.2666	0.5922

Table 2: Comparing bounds in Nielsen and Sandmann (2002) with our results.

σ : yearly volatility
K : strike price
LN : lognormal approximation of a sum of lognormals
IG : inverse gaussian approximation of a sum of lognormals
MC : Monte Carlo price with its standard error ($S.E.$) based on 10 000 paths
LBB_T : lower bound with $\Lambda = \sum_{k=1}^{T} W_k \overset{d}{=} B(T)$
LBFA : lower bound with $\Lambda = \sum_{j=0}^{n-1} \exp[(r - \frac{\sigma^2}{2})(T-j)]B(T-j)$
LBGA : lower bound with $\Lambda = (\ln \mathbb{G} - E[\ln \mathbb{G}])/\sqrt{\text{var}(\ln \mathbb{G})}$
$C_A^{l,Z}$: Nielsen and Sandmann notation for LBGA

UBGA_d : upper bound equal to lower bound LBGA plus $\varepsilon(d_{GA})$
$C_A^{u,Z}$: Nielsen and Sandmann notation for UBGA_d
UBFA_d : upper bound equal to lower bound LBFA plus $\varepsilon(d_{FA})$
UBFA : upper bound equal to lower bound LBFA plus constant $\varepsilon(+\infty)$
RS : Nielsen and Sandmann notation for LBGA plus constant $\varepsilon(+\infty)$
UBB_T : upper bound equal to lower bound LBB_T plus constant $\varepsilon(+\infty)$
PECUB : partially exact/comonotonic upper bound with $\Lambda = (\ln \mathbb{G} - E[\ln \mathbb{G}])/\sqrt{\text{var}(\ln \mathbb{G})}$
ICUB : improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k \overset{d}{=} B(T)$
CUB : comonotonic upper bound
$C_A^{**,G}$: $C_A^{u,G}$ upper bound of Nielsen and Sandmann with optimal weights,
$C_A^{*,G}, C_A^{N,G}$: $C_A^{u,G}$ with special choice for weights, with equal weights

The distance (56) can be interpreted as the distance of the distribution of $\mathbb{S}$ from the closest lognormal distribution in Kullback-Leibler sense. The latter is the distribution which shares the same log-moments $E_p[\ln(\cdot)]^i, i = 1, \ldots, m$ with the distribution of $\mathbb{S}$.
This provides an alternative way to the lognormal approximation of Jacques (1996) in order to compute the price of the Asian call option. Namely, we can estimate the parameters of the closest lognormal distribution based on the simulated $\mathbb{S}$, and then apply the standard Black & Scholes technique in order to find the price. This method is considerably easier to implement than that of Jacques (1996). However, to obtain a correct price approximation, more simulations are needed than for the usual Monte-Carlo price estimate.

In Table 3 we present the results obtained in evaluating the Kullback-Leibler distance for the sum of lognormals $\mathbb{S}$ through a standard Monte Carlo method with $10\,000$ antithetic paths, for the parameters in Table 1. In the brackets we show the sample standard errors (S.E) for both quantities. In order to have an idea for what it means to have a KLI distance of about 0.02 between two distributions, we may resort to the KLI distance of two lognormals, which can be easily computed analytically, see e.g. Brigo and Liinev (2002). It appears that we find a KLI distance comparable in size to our distances above if we consider for example two lognormal densities with the same mean but different standard deviations. Then a KLI distance of approximately 0.02 amounts to a percentage difference in standard deviations of about 1.8%. This gives a feeling for the size of the distributional discrepancy our distance implies.

σ	$\mathbb{S}$ (S.E.)	KLI (S.E)
0.2	$3079.000\,(3.255429)$	$0.0221168\,(0.002064689)$
0.3	$3078.555\,(4.905087)$	$0.0220335\,(0.002086076)$
0.4	$3078.558\,(6.579753)$	$0.0219415\,(0.002109625)$

Table 3: Distance analysis.

In Table 4 we show the corresponding lognormal price approximation. These values seem to indicate that this method performs better than the moment-matching method as presented in Jacques (1996), but still underestimates the price. This indicates that even the optimal lognormal distribution (in KLI sense) does not attribute enough weight to the upper tail.

K	$\sigma = 0.02$	$\sigma = 0.03$	$\sigma = 0.04$
80	22.00133	22.30572	23.02679
90	12.75699	13.91766	15.41261
100	5.515920	7.525337	9.550753
110	1.647747	3.508497	5.504232

Table 4: Price approximation based on the closest lognormal distribution in Kullback-Leibler sense.

In Table 5 we display the Hellinger distances between the densities of $\mathbb{S}^\ell$, (28), when the conditioning variable Λ is given by (25) (hereafter denoted as $\mathbb{S}^\ell_{FA}$), and of the comonotonic sum $\mathbb{S}^c$, (9). It appears that increasing the volatility σ the densities tend to move further away from each other. We also computed the distance between the densities of $\mathbb{S}^\ell_{FA}$ and of $\mathbb{S}^\ell_{GA}$ which is $\mathbb{S}^\ell$ with

σ	$HD(\mathbb{S}^{\ell}_{FA};\mathbb{S}^{c})$
0.2	0.023339455
0.3	0.138331889
0.4	0.312827667

Table 5: Hellinger distance between comonotonic lower and upper bound of $\mathbb{S}$.

conditioning variable Λ (26). This distance was found to be of the magnitude of 10^{-13}, and slightly decreasing with increasing σ.

F. Hedging the fixed strike Asian option

From the analytical expressions for the lower and the upper bounds we can easily obtain the hedging Greeks which are summarized in Table 6. We heavily rely on the Black & Scholes expressions that we found for these bounds.

V Floating strike Asian options in a Black & Scholes settings

The price at current time $t = 0$ of a floating strike Asian put option with percentage β is given by

$$APF(n,\beta,T) = \frac{e^{-rT}}{n} E^{Q}\left[\left(\sum_{i=0}^{n-1} S(T-i) - n\beta S(T)\right)_{+}\right].$$

In the Black & Scholes model, the following change of measure leads to results dealt with in Section III. Let us define the probability $\tilde{Q}$ equivalent to Q by the Radon-Nikodym derivative

$$\frac{d\tilde{Q}}{dQ} = \frac{S(T)}{S(0)e^{rT}} = \exp(-\frac{\sigma^2}{2}T + \sigma B(T)). \tag{57}$$

Under this probability $\tilde{Q}$, $\tilde{B(t)} = B(t) - \sigma t$ is a Brownian motion and therefore, the dynamics of the share under $\tilde{Q}$ are given by

$$\frac{dS(t)}{S(t)} = (r + \sigma^2)dt + \sigma d\tilde{B}(t). \tag{58}$$

Let us first consider the case of a forward starting floating strike Asian option with $T-n+1 > 0$. Using the probability $\tilde{Q}$, the floating strike Asian option with percentage β is given by

$$APF(n,\beta,T) = \frac{S(0)}{n} E^{\tilde{Q}}\left[\left(\frac{\sum_{i=0}^{n-1} S(T-i)}{S(T)} - \beta n\right)_{+}\right].$$

From this formula, one can conjecture that a floating strike Asian put option can be interpreted as a fixed strike Asian call with exercise price $\beta S(0)$. Henderson and Wojakowski (2002) have obtained symmetry results between the floating and fixed strike Asian options in the forward starting case of continuous sampling.

Bound	Delta (Δ)
LBΛ	$\sum_{i=0}^{n-1} \frac{e^{-ri}}{n} \Phi(\rho_{T-i}\sigma\sqrt{T-i} - \Phi^{-1}(F_{\mathbb{S}^\ell}(nK)))$
CUB	$\sum_{i=0}^{n-1} \frac{e^{-ri}}{n} \Phi(\sigma\sqrt{T-i} - \Phi^{-1}(F_{\mathbb{S}^c}(nK)))$
ICUB	$\int_0^1 I_2(v)dv$
PECUB	$\sum_{i=0}^{n-1} \frac{e^{-ri}}{n} \Phi(\rho_{T-i}\sigma\sqrt{t-i} - d^*) + \int_0^{\Phi(d^*)} I_2(v)dv + [\chi(d^*) + I_1(\Phi(d^*))\varphi(d^*)] \frac{\partial d^*}{\partial S(0)}$
UBΛ	$\Delta_{\text{LB}\Lambda} + \frac{e^{-rT}}{2n} \int_0^1 \sqrt{q(v)}dv$
UBΛ_d	$\Delta_{\text{LB}\Lambda} + g(d^*)[1+\eta(d^*)]$
	Gamma (Γ)
LBΛ	$e^{-rT} n \left[\frac{K}{S(0)}\right]^2 \varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(nK))) \left[\sum_{i=0}^{n-1} \tilde{K}_i \sigma \rho_{T-i}\sqrt{T-i}\right]^{-1}$
CUB	$e^{-rT} n \left[\frac{K}{S(0)}\right]^2 \varphi(\Phi^{-1}(F_{\mathbb{S}^c}(nK))) \left[\sum_{i=0}^{n-1} K_i \sigma \sqrt{T-i}\right]^{-1}$
ICUB	$e^{-rT} n \left[\frac{K}{S(0)}\right]^2 \int_0^1 \varphi(\Phi^{-1}(F_{\mathbb{S}^u\|V=v}(nK))) \left[\sum_{i=0}^{n-1} \tilde{K}_i(v) p_i(v) e^{-r(T-i)} \sigma \sqrt{1-\rho_{T-i}^2}\sqrt{T-i}\right]^{-1} dv$
PECUB	$\left[-\frac{e^{-rT}}{n} \sum_{i=0}^{n-1} p_i(\Phi(d^*)) + 2I_2(\Phi(d^*)) + \frac{\partial \chi(d^*)}{\partial S(0)}\right] \varphi(d^*) \frac{\partial d^*}{\partial S(0)} + \chi(d^*) \frac{\partial^2 d^*}{\partial S(0)^2} + \gamma(d^*) I_1(\Phi(d^*))$ $+e^{-rT} n \left[\frac{K}{S(0)}\right]^2 \int_0^{\Phi(d^*)} \varphi(\Phi^{-1}(F_{\mathbb{S}^u\|V=v}(nK))) \left[\sum_{i=0}^{n-1} \tilde{K}_i(v) p_i(v) e^{-r(T-i)} \sigma \sqrt{1-\rho_{T-i}^2}\sqrt{T-i}\right]^{-1} dv$
UBΛ	$\Gamma_{\text{LB}\Lambda}$
UBΛ_d	$\Gamma_{\text{LB}\Lambda} + \frac{1}{2} g(d^*) \left[(1+\eta(d^*))\eta(d^*) + \frac{\partial \eta(d^*)}{\partial d^*}\right] \frac{\partial d^*}{\partial S(0)}$
	Vega ($\mathcal{V}$)
LBΛ	$\frac{e^{-rT}}{n} \left[\sum_{i=0}^{n-1} \tilde{K}_i \rho_{T-i}\sqrt{T-i}\right] \varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(nK)))$
CUB	$\frac{e^{-rT}}{n} \left[\sum_{i=0}^{n-1} K_i \sqrt{T-i}\right] \varphi(\Phi^{-1}(F_{\mathbb{S}^c}(nK)))$
ICUB	$\int_0^1 \frac{\partial I_1(v)}{\partial \sigma} dv$
PECUB	$[\chi(d^*) + I_1(\Phi(d^*))\varphi(d^*)] \frac{\partial d^*}{\partial \sigma} + \frac{e^{-rT}}{n} S(0)\varphi(d^*) \sum_{i=0}^{n-1} p_i(\Phi(d^*)) \rho_{T-i}\sqrt{T-i} + \int_0^{\Phi(d^*)} \frac{\partial I_1(v)}{\partial \sigma} dv$
UBΛ	$\mathcal{V}_{\text{LB}\Lambda} + \frac{e^{-rT}}{4n} S(0) \int_0^1 \frac{1}{\sqrt{q(v)}} \frac{\partial q(v)}{\partial \sigma} dv$
UBΛ_d	$\mathcal{V}_{\text{LB}\Lambda} + \frac{S(0)}{2} g(d^*)\, \zeta(d^*) \frac{\partial d^*}{\partial \sigma}$

Notations

$$p_i(v) = e^{(r - \frac{\sigma^2}{2}\rho_{T-i}^2)(T-i)} e^{\rho_{T-i}\sigma\sqrt{T-i}\,\Phi^{-1}(v)}$$

$$I_1(v) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{p_i(v)}{e^{r(T-i)}} \left[e^{-ri} S(0) \Phi(d_{1,i}(v)) - e^{-rT} \tilde{K}_i(v) \Phi(d_{2,i}(v))\right]$$

$$I_2(v) = \frac{\partial I_1(v)}{\partial S(0)} = \frac{e^{-rT}}{n} \sum_{i=0}^{n-1} p_i(v) \Phi(d_{1,i}(v))$$

$$c_{ij} = e^{\sigma^2 (\min(T-i, T-j) - \rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j})}$$

$$q_{ij} = e^{r(2T-i-j) + \sigma^2 \rho_{T-i}\rho_{T-j}\sqrt{T-i}\sqrt{T-j}} (c_{ij} - 1)$$

$$h(d^*) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q_{ij} \Phi(d^* - \sigma(\rho_{T-i}\sqrt{T-i} + \rho_{T-j}\sqrt{T-j}))$$

$$g(d^*) = \frac{e^{-rT}}{2n} \Phi(d^*)^{1/2} h(d^*)^{1/2}$$

$$\zeta(d^*) = \frac{\varphi(d^*)}{\Phi(d^*)} + \frac{1}{h(d^*)} \frac{\partial h(d^*)}{\partial d^*}$$

$$\eta(d^*) = \frac{S(0)}{2} \zeta(d^*) \frac{\partial d^*}{\partial S(0)}$$

$$\gamma(d^*) = \varphi(d^*) \left[-d^* (\frac{\partial d^*}{\partial S(0)})^2 + \frac{\partial^2 d^*}{\partial S(0)^2}\right]$$

$$\chi(d^*) = e^{-rT} \varphi(d^*) \left[K - \frac{S(0)}{n} \sum_{i=0}^{n-1} p_i(\Phi(d^*))\right]$$

$$q(v) = \frac{\text{var}(\mathbb{S}|\Lambda)}{S(0)} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} p_i(v) p_j(v) c_{ij} - \left[\sum_{i=0}^{n-1} p_i(v)\right]^2$$

Table 6: Delta, Gamma and Vega for bounds.

In Section F., we prove similar results in case of the discretely sampled Asian options. The symmetry results become very useful for transferring knowledge about one type of an option to another. Writing down the formulae for $S(T-i)$ and $S(T)$ in the Black & Scholes setting leads to

$$\mathbb{S} = \frac{\sum_{i=0}^{n-1} S(T-i)}{S(T)} = \sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2})i+\sigma(\tilde{B}(T-i)-\tilde{B}(T))} \overset{\text{not}}{=} \sum_{i=0}^{n-1} \alpha_i \, e^{Y_i}$$

with $\alpha_i = e^{-(r+\frac{\sigma^2}{2})i}$ and with $Y_i = \sigma(\tilde{B}(T-i) - \tilde{B}(T))$ a normally distributed random variable with mean $E^{\tilde{Q}}[Y_i] = 0$ and variance $\sigma_{Y_i}^2 = i\sigma^2$. Note that $\alpha_0 e^{Y_0}$ is in fact a constant. Clearly $\mathbb{S}$ is a sum of lognormal variables and thus we can apply the results of Section III.

Denoting the price of an arithmetic floating strike European-style Asian *call* option with exercise date T, n averaging dates and percentage β by $ACF(n,\beta,T)$, we find from the put-call parity at the present:

$$APF(n,\beta,T) - ACF(n,\beta,T) = \frac{S(0)}{n}\frac{1-e^{-rn}}{1-e^{-r}} - \beta S(0). \tag{59}$$

Hence, we can derive bounds for the Asian floating strike call option from the bounds for the put.

In the remaining of the section, we only work out in detail the forward starting case as the 'in progress' case can be dealt with in a similar way.

A. Lower bound

In order to obtain a lower bound of good quality for the forward starting Asian option, we consider as conditioning variable a normal random variable Λ which is as much alike as $\mathbb{S}$. Inspired by the choice for the fixed case, we take

$$\Lambda = \sum_{i=0}^{n-1} \beta_i (\tilde{B}(T-i) - \tilde{B}(T)) \tag{60}$$

with β_i some positive reals. In particular for $\beta_i = e^{-(r+\frac{\sigma^2}{2})i}$ we find the first order approximation of $\mathbb{S}$. If β_i equals $\frac{1}{\sqrt{\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n}}$ for all i, then $\Lambda = \frac{\ln \mathbb{G} - E[\ln \mathbb{G}]}{\sqrt{\text{var}[\ln \mathbb{G}]}}$ is the standardized logarithm of the geometric average $\mathbb{G}$:

$$\mathbb{G} = \left(\prod_{i=0}^{n-1} \frac{S(T-i)}{S(T)}\right)^{1/n} = \left(\prod_{i=0}^{n-1} \exp\left[-(r+\frac{\sigma^2}{2})i + \sigma(\tilde{B}(T-i) - \tilde{B}(T))\right]\right)^{1/n}, \tag{61}$$

with

$$\begin{aligned} E^{\tilde{Q}}[\ln \mathbb{G}] &= -(r+\frac{\sigma^2}{2})\frac{n-1}{2} \\ \text{var}[\ln \mathbb{G}] &= \frac{\sigma^2}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\min(i,j) = \frac{\sigma^2}{n^2}\left(\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n\right). \end{aligned}$$

This choice of Λ is similar to the choice (26) of Nielsen and Sandmann (2002) in the fixed strike setting.
For general β_i, we have that $Y_i \mid \Lambda = \lambda$ is normally distributed with mean $r_i \frac{\sigma\sqrt{i}}{\sigma_\Lambda}\lambda$ and variance $\sigma^2_{Y_i}\left(1 - r_i^2\right)$ where $r_0 = 0$ and for $i \geq 1$

$$r_i = \frac{\operatorname{cov}\left(\tilde{B}(T-i) - \tilde{B}(T), \Lambda\right)}{\sqrt{i}\,\sigma_\Lambda} = \frac{\sum_{j=0}^{n-1} \beta_j \min(i,j)}{\sqrt{i}\sqrt{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \beta_i\beta_j \min(i,j)}}. \tag{62}$$

For both choices of Λ that we consider, these correlations r_i are positive. We thus find from (21) the following lower bound for the price of the forward starting Asian floating put option:

$$APF(n,\beta,T) \geq \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi\left[\sigma r_i\sqrt{i} - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(n\beta)\right)\right] - S(0)\beta\left(1 - F_{\mathbb{S}^\ell}(n\beta)\right),$$

where $F_{\mathbb{S}^\ell}(n\beta)$ is obtained from (20) for $x = n\beta$.

B. Comonotonic upper bound

Applying (13) we get the following explicit formula of the comonotonic upper bound for the forward starting Asian floating strike options:

$$APF(n,\beta,T) \leq \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi\left[\sigma\sqrt{i} - \Phi^{-1}\left(F_{\mathbb{S}^c}(n\beta)\right)\right] - S(0)\beta\left(1 - F_{\mathbb{S}^c}(n\beta)\right),$$

which holds for any $\beta > 0$ and where $F_{\mathbb{S}^c}(n\beta)$ follows from (12) for $x = n\beta$.

C. Improved comonotonic upper bound

Analogous to the case of the improved upper bound for the Asian fixed strike, we have found that also in the Asian floating strike case, the conditioning variable

$$\Lambda = -\sum_{k=1}^{T} W_k, \quad \text{with } W_k \text{ i.i.d. } N(0,1) \text{ such that } \tilde{B}(T-i) \stackrel{d}{=} \sum_{k=1}^{T-i} W_k,\ i = 0,\ldots,n-1,$$

leads to a sharper upper bound than other choices, for example the conditioning variable in the lower bound.
The theory of comonotonicity (see (19) and (17)) then leads to the following upper bound

$$\begin{aligned}
&\frac{S(0)}{n}E^{\tilde{Q}}\left[\left(\mathbb{S}^u - n\beta\right)_+\right] \\
&= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-(r+\frac{\sigma^2}{2}r_i^2)i}\int_0^1 e^{r_i\sigma\sqrt{i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-r_i^2}\,\sigma\sqrt{i} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(n\beta)\right)\right)dv \\
&\quad - S(0)\beta\left(1 - F_{\mathbb{S}^u}(n\beta)\right)
\end{aligned}$$

with the correlations given by $r_i = \sqrt{\frac{i}{T}}$, $i = 1,\ldots,n-1$ and $r_0 = 0$. Invoking (15)-(16), the conditional distribution $F_{\mathbb{S}^u|V=v}(x)$ and the cdf of $\mathbb{S}^u$ can be obtained.

D. Bounds based on the Rogers & Shi approach

By a similar reasoning as in Section IV.B., it is easy to derive an upper bound based on the lower bound by following the ideas of Rogers and Shi (1995) and Nielsen and Sandmann (2002). Indeed, by using our conditioning variable Λ given by (60), we obtain

$$APF(n,\beta,T) \leq \frac{S(0)}{n}\left\{E^{\tilde{Q}}\left[(\mathbb{S}^{\ell}-n\beta)^{+}\right]+\varepsilon(d)\right\}$$

where d is such that $\mathbb{S}\geq n\beta$ if $\Lambda\geq d$ and with

$$\varepsilon(d)=\frac{1}{2}\left\{F_{\Lambda}(d)\right\}^{\frac{1}{2}}\times$$
$$\times\left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}e^{-r(i+j)+\sigma^2 r_i r_j\sqrt{i}\sqrt{j}}\Phi\left(d^*-\sigma(r_i\sqrt{i}+r_j\sqrt{j})\right)\left(e^{\sigma^2\left(\min(i,j)-r_i r_j\sqrt{i}\sqrt{j}\right)}-1\right)\right\}^{\frac{1}{2}}$$

where $d^*=\frac{d-E^{\tilde{Q}}[\Lambda]}{\sigma_\Lambda}$ and $F_\Lambda(\cdot)$ stands for the normal cumulative distribution function of Λ, and with correlations r_i defined in (62).

In particular for the linear transformation of the first order approximation (FA) of $\mathbb{S}$, namely $\Lambda=\sum_{i=0}^{n-1}\beta_i(\tilde{B}(T-i)-\tilde{B}(T))$ with $\beta_i=e^{-(r+\frac{\sigma^2}{2})i}$,

$$\tilde{d}_{FA}=\frac{n\beta-\sum_{i=0}^{n-1}e^{-(r+\frac{\sigma^2}{2})i}}{\sigma}.$$

For $\beta_i=\frac{n}{\sigma}\mathrm{Var}[\ln\mathbb{G}]$ with the geometric average (GA) $\mathbb{G}$ defined in (61), Λ equals the standardized logarithm of the geometric average and the corresponding d equals

$$\tilde{d}_{GA}=\frac{\ln(\beta)+(r+\frac{\sigma^2}{2})\frac{n-1}{2}}{\frac{\sigma}{n}\sqrt{\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n}}.$$

E. Partially exact/comonotonic upper bound

Along similar lines as in Section IV.C., we can derive a partially exact/comonotonic upper bound by recalling that for some normally distributed variable Λ there exists a d such that $\Lambda\geq d$ implies $\mathbb{S}\geq n\beta$:

$$\frac{S(0)}{n}\sum_{i=0}^{n-1}e^{-ri}\Phi(r_i\sigma\sqrt{i}-d^*)-S(0)\beta\Phi(-d^*)$$
$$+\frac{S(0)}{n}\sum_{i=0}^{n-1}e^{-(r+\frac{\sigma^2}{2}r_i^2)i}\int_0^{\Phi(d^*)}e^{r_i\,\sigma\sqrt{i}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-r_i^2}\,\sigma\sqrt{i}-\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(n\beta)\right)\right)dv$$
$$-S(0)\beta\left(\Phi(d^*)-\int_0^{\Phi(d^*)}F_{\mathbb{S}^u|V=v}(n\beta)\,dv\right)$$

where $d^* = \dfrac{d - E^{\tilde{Q}}[\Lambda]}{\sigma_\Lambda}$ and $v = \dfrac{\lambda - E^{\tilde{Q}}[\Lambda]}{\sigma_\Lambda}$.

The first two terms are composing the exact part of $\frac{S(0)}{n} E^{\tilde{Q}}[(\mathbb{S} - n\beta)_+]$, while the last two terms define the improved comonotonic upper bound for the remaining part of it.

F. Symmetry results for arithmetic Asian options

Henderson and Wojakowski (2002) have obtained symmetry results between the floating and fixed strike Asian options in the forward starting case of continuous sampling. They considered the Black & Scholes dynamics for the underlying asset with a continuous dividend yield δ. In order to generalize their results, we introduce some generalized notation. In particular, $ACF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0)$ denotes the European-style floating strike Asian call option with percentage $\frac{K}{S(0)}$ and maturity date T which is forward starting with n terms and with the first term being $S(0)$, where $(S(t))_t$ denotes as usual a Black & Scholes process with initial value $S(0)$ and with dividend yield r. The short-term constant interest rate equals δ.

For fixed strike options, we introduce a similar notation. Let $AP(K, S(0), r, \delta, T, n, T-n+1)$ be a fixed strike Asian put with fixed exercise price K and maturity date T which is forward starting with n terms and with the first term being $S(T-n+1)$, where $(S(t))_t$ denotes as usual a Black & Scholes process with initial value $S(0)$ and with dividend yield δ. The short-term constant interest rate equals r.

Using these notations, we obtain the following symmetry results, which are proved in the Appendix.

Theorem 3.

$$\begin{aligned} AP(K, S(0), r, \delta, T, n, T-n+1) &= ACF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0) \\ ACF(S(0), \beta, r, \delta, T, n, T-n+1) &= AP(\beta S(0), S(0), \delta, r, T, n, 0) \end{aligned}$$

and

$$\begin{aligned} AC(K, S(0), r, \delta, T, n, T-n+1) &= APF(S(0), \frac{K}{S(0)}, \delta, r, T, n, 0) \\ APF(S(0), \beta, r, \delta, T, n, T-n+1) &= AC(\beta S(0), S(0), \delta, r, T, n, 0). \end{aligned}$$

G. Numerical illustration

In this section we shall give a numerical example of a floating strike Asian put option.

In Table 7 we display different lower and upper bounds for a floating strike Asian put option with an initial stock price $S(0) = 100$, a maturity of 120 days and an averaging period n of 30 days. The choices for volatility and risk-free interest rate are the same as in Section IV.E. The percentage β is chosen so that $\beta S(0)$ corresponds to the respective strike K in Section IV.E. We obtained Monte Carlo price estimates (based on $10\,000$ simulated paths) by adapting the Kemna and Vorst (1990) control variate technique. Indeed, by applying the change of measure (57), we can interpret a floating strike Asian put option as a fixed strike Asian call option with exercise

price $\beta S(0)$. Hence we can simulate the dynamics of the stock price according to (58), and use the geometric average $\mathbb{G}$ given by (61) as our control variate.
Note that by using the put-call parity result (59) one can easily obtain the price for the floating strike Asian call option. For example, consider the entry in Table 7 with $\beta = 1.0$, $\sigma = 0.2$, and $r = 0.05$. By applying (59), we obtain, for instance, that LB$FA = 1.387410$, LB$GA = 1.387411$, UB$GA_d = 1.388847$, UB$FA_d = 1.388792$, PECUB $=1.557532$, ICUB $=1.575395$, and CUB $=1.583292$.

$r = 0.09$

σ	β	MC (S.E.(10^{-4}))	LBFA	LBGA	UBGA_d	UBFA_d	UBFA	PECUB	ICUB	CUB
0.2	0.8	19.64351 (2.5)	19.643331	19.643331	19.643331	19.643331	19.652053	19.643118	19.643284	19.643331
	0.9	9.644117 (2.5)	9.643903	9.643903	9.643923	9.643934	9.652625	9.645429	9.646147	9.646371
	1.0	1.113866 (2.1)	1.113997	1.113998	1.119154	1.118720	1.122719	1.283311	1.301119	1.308984
	1.1	0.001167 (0.6)	0.001154	0.001155	0.010306	0.010293	0.009876	0.004286	0.004762	0.005027
0.3	0.8	19.64376 (5.6)	19.643332	19.643332	19.643333	19.643334	19.662815	19.642851	19.643255	19.643365
	0.9	9.670844 (5.3)	9.670327	9.670324	9.671056	9.671175	9.689810	9.704453	9.708673	9.710906
	1.0	1.752636 (5.0)	1.753406	1.753406	1.764434	1.763671	1.772889	2.008637	2.034843	2.046686
	1.1	0.040901 (3.2)	0.040840	0.040844	0.060394	0.060568	0.060323	0.084571	0.089851	0.092513
0.4	0.8	19.64452 (9.9)	19.643666	19.643666	19.643700	19.643762	19.678319	19.645280	19.645424	19.645780
	0.9	9.784575 (9.1)	9.784545	9.784533	9.788040	9.788243	9.819198	9.891788	9.904717	9.910791
	1.0	2.391664 (9.1)	2.393883	2.393884	2.412935	2.411692	2.428536	2.734542	2.769381	2.785206
	1.1	0.191081 (7.4)	0.192114	0.192128	0.224217	0.224551	0.226767	0.320139	0.334277	0.341203

$r = 0.05$

σ	β	MC (S.E.(10^{-4}))	LBFA	LBGA	UBGA_d	UBFA_d	UBFA	PECUB	ICUB	CUB
0.2	0.8	19.80180 (2.5)	19.801637	19.801637	19.801637	19.801637	19.810313	19.801423	19.801590	19.801637
	0.9	9.802297 (2.5)	9.802114	9.802114	9.802131	9.8021409	9.810790	9.803394	9.804074	9.804279
	1.0	1.188935 (2.2)	1.189061	1.189061	1.193931	1.1936643	1.197736	1.359169	1.377045	1.384942
	1.1	0.001407 (0.7)	0.001377	0.001377	0.010502	0.0105246	0.010052	0.004943	0.005479	0.005775
0.3	0.8	19.80200 (5.6)	19.801638	19.801638	19.801638	19.801640	19.821132	19.801156	19.801557	19.801667
	0.9	9.826784 (5.3)	9.826301	9.826299	9.826970	9.8271007	9.845795	9.858436	9.862434	9.864562
	1.0	1.830198 (5.1)	1.830953	1.830953	1.841571	1.8410460	1.850447	2.086848	2.113107	2.124978
	1.1	0.044667 (3.3)	0.044669	0.044671	0.064136	0.0643551	0.064163	0.091056	0.096617	0.099416
0.4	0.8	19.80267 (10.0)	19.801942	19.801942	19.801972	19.802032	19.836644	19.803444	19.803566	19.803912
	0.9	9.935006 (9.2)	9.935044	9.935035	9.938357	9.9386211	9.969747	10.038765	10.051296	10.057206
	1.0	2.470755 (9.2)	2.473011	2.473011	2.491532	2.4905981	2.507713	2.814307	2.849193	2.865046
	1.1	0.202340 (7.6)	0.203494	0.203505	0.235379	0.2357953	0.238196	0.335905	0.350466	0.357594

Table 7: Comparing bounds for a floating strike Asian put option

$T = 120, n = 30,\ \sigma:$ yearly volatility, $\beta:$ percentage, $S(0) = 100$
MC : Monte Carlo price together with its standard error ($S.E.$) based on 10 000 paths
LBFA : lower bound with $\Lambda = \sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-j)} B(T-j)$
LBGA : lower bound with $\Lambda = (\ln \mathbb{G} - E[\ln \mathbb{G}])/\sqrt{\text{var}(\ln \mathbb{G})}$
UBGA_d : upper bound equal to lower bound LBGA plus $\varepsilon(d_{GA})$
UBFA_d : upper bound equal to lower bound LBFA plus $\varepsilon(d_{FA})$
UBFA : upper bound equal to lower bound LBFA plus constant $\varepsilon(+\infty)$
PECUB : partially exact/comonotonic upper bound with $\Lambda = (\ln \mathbb{G} - E[\ln \mathbb{G}])/\sqrt{\text{var}(\ln \mathbb{G})}$
ICUB : improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k \stackrel{d}{=} B(T)$
CUB : comonotonic upper bound

We observe similar behaviour of the bounds as for the fixed strike Asian option apart from some interesting particular cases:

1. for $\sigma = 0.2, 0.3, 0.4$ and $\beta = 0.8$ the lower and the best upper bounds coincide up to three or four decimals and thus give almost exact results;

2. for $\sigma = 0.2$ and 0.3, and $\beta = 1.1$ the value for upper bound UBFA_d is larger than the one for UBFA which must be caused by the additional Hölder inequality in the derivation of the error bound $\varepsilon(\tilde{d}_{FA})$;

3. the partially exact/comonotonic upper bound PECUB is always smaller than ICUB and is even the best of all upper bounds for $\sigma = 0.2$ and $\beta = 1.1$.

Note that for $\beta = 0.8$ ($\sigma = 0.2, 0.3$) — which is a case of theoretical interest — the values of PECUB and ICUB suffer from numerical instabilities caused by the involved numerical integration.

VI Conclusions

We derived accurate lower and upper bounds for the price of discretely sampled European-style arithmetic Asian options with fixed and floating strike. Hereto we used and combined different ideas and techniques such as firstly conditioning on some random variable as in Rogers and Shi (1995), secondly results based on comonotonic risks and bounds for stop-loss premiums of sums of dependent random variables as in Kaas, Dhaene and Goovaerts (2000), and finally adaptation of the error bound of Rogers and Shi as in Nielsen and Sandmann (2002). All bounds have analytical expressions. This allows a study of the hedging Greeks of these bounds. For the numerical experiments it was important to find and motivate a good choice for the conditioning variables appearing in the formulae. We note that the expressions found for the bounds are not only analytical but also easily computable. The numerical results in the tables show that the upper bound UBGA_d is in general the best one except for extreme values of the strike price K or β; then ICUB or PECUB outperforms all the other upper bounds.

In a forthcoming paper, we use this approach to derive upper and lower bounds for basket options and Asian basket options. We also concentrate on the derivation of bounds for Asian options by using binomial trees.
Further, we plan to derive bounds for the Asian options in case the underlying follows mixed jump-diffusion dynamics. These results are in particular interesting for pricing catastrophe insurance derivatives as Cat-calls (see Geman (1994)).

VII Acknowledgments

G. Deelstra, J. Liinev and M. Vanmaele would like to acknowledge the financial support by the BOF-project 001104599 of the Ghent University.
Jan Dhaene and Marc Goovaerts would like to acknowledge the financial support of the Onderzoeksfonds K.U. Leuven (GOA/02: Actuariële, financiële en statistische aspecten van afhankelijkheden in verzekerings- en financiële portefeuilles).

Appendix A. Proof of symmetry results in Theorem 3

Proof. We only prove the first symmetry result since the others follow along similar lines.

$$
\begin{aligned}
&AP(K,S(0),r,\delta,T,n,T-n+1)\\
&\quad = e^{-rT}E^{Q}\left[\left(K-\frac{1}{n}\sum_{i=0}^{n-1}S(T-i)\right)_+\right]\\
&\quad = e^{-\delta T}E^{Q}\left[\frac{e^{-(r-\delta)T}S(T)}{S(0)}\left(\frac{KS(0)}{S(T)}-\frac{1}{n}\sum_{i=0}^{n-1}\frac{S(T-i)S(0)}{S(T)}\right)_+\right]\\
&\quad = e^{-\delta T}E^{\tilde{Q}}\left[\left(\frac{KS(0)}{S(T)}-\frac{1}{n}\sum_{i=0}^{n-1}S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i+\sigma\left(\tilde{B}(T-i)-\tilde{B}(T)\right)\right]\right)_+\right],
\end{aligned}
$$

where we defined as before the probability $\tilde{Q}$ equivalent to Q by the Radon-Nikodym derivative but now by stressing the dividend yield δ

$$
\frac{d\tilde{Q}}{dQ}=\frac{S(T)}{S(0)e^{(r-\delta)T}}=\exp(-\frac{\sigma^2}{2}T+\sigma B(T)).
$$

Under this probability $\tilde{Q}$, $\tilde{B}(t)=B(t)-\sigma t$ is a Brownian motion and therefore, the dynamics of the share under $\tilde{Q}$ are given by

$$
\frac{dS(t)}{S(t)}=((r-\delta)+\sigma^2)dt+\sigma d\tilde{B}(t).
$$

Due to the independent increments, $\tilde{B}(T-i)-\tilde{B}(T)$ has the same distribution as $\tilde{B}(i)$ and $-\tilde{B}(i)$, and we can concentrate on the process $(S^*(t))_t$ defined by

$$
S^*(i)=S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i+\sigma\tilde{B}(i)\right].
$$

Indeed, then

$$
\begin{aligned}
AP(K,S(0),r,\delta,T,n,T-n+1) &= e^{-\delta T}E^{\tilde{Q}}\left[\left(\frac{KS^*(T)}{S(0)}-\frac{1}{n}\sum_{i=0}^{n-1}S^*(i)\right)_+\right]\\
&= e^{-\delta T}E^{Q}\left[\left(\frac{K\tilde{S}(T)}{S(0)}-\frac{1}{n}\sum_{i=0}^{n-1}\tilde{S}(i)\right)_+\right]
\end{aligned}
$$

with the process $(\tilde{S}(t))_t$ defined by

$$
\tilde{S}(i)=S(0)\exp\left[-(r-\delta+\frac{\sigma^2}{2})i+\sigma B(i)\right]
$$

with $(B(t))_t$ a Brownian motion under Q.
As a conclusion,

$$
AP(K,S(0),r,\delta,T,n,T-n+1)=ACF(S(0),\frac{K}{S(0)},\delta,r,T,n,0),
$$

which was to be shown. ♢

References

[1] Black, F., and Scholes, M. “The pricing of options and corporate liabilities”, *Journal of Political Economy*, **81** (1973), 637-659.

[2] Brigo, D., and Liinev, J. “On the distributional distance between the Libor and the Swap market models”, *Preprint*, (2002).

[3] Curran, M. “Valuing Asian and portfolio options by conditioning on the geometric mean price”, *Management Science*, **40**(12) (1994), 1705-1711.

[4] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., and Vyncke, D. “The concept of comonotonicity in actuarial science and finance: theory”, forthcoming in *Insurance: Mathematics & Economics*, (2002).

[5] Geman, H. “CAT-calls”, *Risk*, **7**(9) (1994), 86-89.

[6] Henderson, V., and Wojakowski, R. “On the equivalence of floating and fixed-strike Asian options”, *Journal of Applied Probability*, **39**(2) (2002), 391-394.

[7] Jacques, M. “On the Hedging Portfolio of Asian Options”, *ASTIN Bulletin*, **26** (1996), 165-183.

[8] Kaas, R., Dhaene, J., and Goovaerts, M.J. “Upper and lower bounds for sums of random variables”, *Insurance: Mathematics & Economics*, **27** (2000), 151-168.

[9] Kemna, A.G.Z., and Vorst, A.C.F. “A pricing method for options based on average asset values”, *Journal of Banking and Finance*, **14** (1990), 113-129.

[10] Klassen, T.R. “Simple, fast, and flexible pricing of Asian options”, *Journal of Computational Finance*, **4**(3) (2001), 89–124.

[11] Nielsen, J.A., and Sandmann, K. “Pricing Bounds on Asian options”, forthcoming in *Journal of Financial and Quantitative Analysis*, (2002).

[12] Rogers, L.C.G., and Shi, Z. “The value of an Asian option”, *Journal of Applied Probability* , **32** (1995), 1077-1088.

[13] Simon, S., Goovaerts, M.J., and Dhaene, J. “An easy computable upper bound for the price of an arithmetic Asian option”, *Insurance: Mathematics & Economics*, **26**(2-3) (2000), 175-184.

[14] Večeř, J. “A new PDE approach for pricing arithmetic average Asian options”, *Journal of Computational Finance*, **4**(4) (2001), 105–113.