Working paper version (23 December 2002)

BOUNDS FOR THE PRICE OF ARITHMETIC BASKET AND ASIAN BASKET OPTIONS

G. DEELSTRA[†], J. LIINEV [*] AND
M. VANMAELE,[*] [**] *Ghent University*

Abstract

An (Asian) basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets (stocks). Determining the price of the basket option is not a trivial task, because in general there is no explicit analytical expression available for the distribution of the weighted sum of the assets.
We derive analytical lower and upper bounds by using on one hand the method of conditioning as in Rogers and Shi (1995), and on the other hand results on a general technique based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables (see Kaas, Dhaene and Goovaerts (2000)).

Keywords: basket option; comonotonicity; analytical bounds; Asian basket option

AMS 2000 Subject Classification: Primary 91B28
Secondary 60E15;60J65

1. Introduction

One of the more extensively sold exotic options is the basket option, an option whose payoff depends on the value of a portfolio or basket of assets. At maturity it pays off the greater of zero and the difference between the average of the prices of the n different assets in the basket and the exercise price.

The typical underlying of a basket option is a basket consisting of several stocks, that represents a certain economy sector, industry or region.

The main advantage of a basket option is that it is cheaper to use a basket option for portfolio insurance than to use the corresponding portfolio of plain vanilla options. Indeed, a basket option takes the imperfect correlation between the assets in the basket into account and moreover the transaction costs are minimized because an investor has to buy just one option instead of several ones.

For pricing simple options on one underlying the financial world has generally adopted the celebrated Black & Scholes model, which leads to a closed form solution for simple options since the stock price at a fixed time follows a lognormal distribution. However, using the famous Black & Scholes model for a collection of underlying stocks, does not provide us with a closed form solution for the price of a basket option.

[*] Postal address: Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281, building S9, 9000 Gent, Belgium
[**] Email address: Michele.Vanmaele@rug.ac.be
[†] Current address: Department of Mathematics/ISRO/ECARES Université Libre de Bruxelles CP210, 1050 Brussels, Belgium

The difficulty stems primarily from the lack of availability of the distribution of a weighted average of lognormals, a feature that has hampered closed-form basket option pricing characterization. Indeed, the value of a portfolio is the weighted average of the underlying stocks at the exercise date.

One can use Monte Carlo simulation techniques (by assuming that the assets follow correlated geometric Brownian motion processes) to obtain a numerical estimate of the price. Other techniques consist of approximating the real distribution of the payoffs by another more tractable one. For instance, finance people use since ages the lognormal distribution as an approximation for the sum of lognormals, although it is common knowledge that this methodology leads sometimes to poor results. An extensive discussion of different methods can be found in the theses of Arts (1999), Beißer (2001) and Van Diepen (2002).

Obviously, the payoff structure of a basket option resembles the payoff structure of an Asian option. But whereas the Asian option is a path-dependent option, that is, its payoff at maturity depends on the price process of the underlying asset, the basket option is a path-independent option whose terminal payoff is a function of several asset prices at the maturity date. Nevertheless, in literature, different authors tried out initial methods for Asian options to the case of basket options. In this respect, it seems natural to adapt the methods developed in Vanmaele et al. (2002) for valuing Asian options and indeed, we have transferred them in a promising way to basket options.

Combining both types of options one can consider an Asian option on a basket of assets instead of on one single asset. In this case we talk about an Asian basket option. Dahl and Benth (2001a,b) value such options by quasi-Monte Carlo techniques and singular value decomposition.

But as these approaches are rather time consuming, it would be vital to have accurate, analytical and easily computable bounds of this price. As the financial institutions dealing with baskets are perhaps even more concerned about the ability of controlling the risks involved, it is important to offer an interval of hedge parameters. Confronted with such issues, the objective of this paper is to obtain accurate analytical lower and upper bounds. To this end, we use on one hand the method of conditioning as in Rogers and Shi (1995), and on the other hand results on a general technique based on comonotonic risks for deriving upper and lower bounds for stop-loss premiums of sums of dependent random variables (see Kaas, Dhaene and Goovaerts (2000)).
All lower and upper bounds can be expressed as an average of Black & Scholes option prices, sometimes with a synthetic underlying asset. Therefore, hedging parameters can be obtained in a straightforward way.

A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets (stocks). Thus, an arithmetic basket call option with exercise date T, n risky assets and exercise price K generates a payoff $\left(\sum_{i=1}^n a_i S_i(T) - K\right)_+$ at T, that is, if the sum $\mathbb{S} = \sum_{i=1}^n a_i S_i(T)$ of asset prices S_i weighted by positive constants a_i at date T is more than K, the payoff equals the difference; if not, the payoff is zero. The price of the basket option at current time $t = 0$ is given by

$$BC(n, K, T) = e^{-rT} E^Q \left[\left(\sum_{i=1}^n a_i S_i(T) - K \right)_+ \right] \tag{1}$$

under a martingale measure Q and with r the risk-neutral interest rate.

Assuming a Black & Scholes setting, the random variables $S_i(T)/S_i(0)$ are lognormally distributed under the unique risk-neutral measure Q with parameters $(r - \sigma_i^2/2)T$ and $\sigma_i^2 T$, when σ_i is the volatility of the underlying risky asset S_i. Therefore we do not have an explicit analytical expression for the distribution of the sum $\sum_{i=1}^n a_i S_i(T)$ and determining the price of the basket option is not a trivial task. Since the problem of pricing arithmetic basket options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent risks, we can apply the results on comonotonic upper and lower bounds for stop-loss premiums, which have been summarized in Section 2.

The paper is organized as follows. Section 2 recalls from Kaas et al. (2000) procedures for obtaining the lower and upper bounds for prices by using the notion of comonotonicity. Section 3 provides bounds for basket options in the Black & Scholes setting, first by concentrating on the comonotonicity and then by applying the Rogers and Shi approach to carefully chosen conditioning variables. We also provide upper bounds by generalizing the Nielsen and Sandmann (2002) idea and by combining it with the notion of comonotonicity. We discuss different conditioning variables in order to determine some superiority. Section 4 contains some general remarks. In Section 5, several sets of numerical results are given and the different bounds are discussed. In particular the correlation among the different underlying assets plays an important role for determining the sharpest price-intervals. Section 6 discusses the pricing of Asian basket options, which can be done by the same reasoning. Section 7 concludes the paper.

2. Some theoretical results

In this section, we recall from Dhaene et al. (2002) and the references therein the procedures for obtaining the lower and upper bounds for stop-loss premiums of sums $\mathbb{S}$ of dependent random variables by using the notion of comonotonicity. A random vector $(X_1^c, \ldots, X_n^c)$ is *comonotonic* if each two possible outcomes $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $(X_1^c, \ldots, X_n^c)$ are ordered componentwise.

In both financial and actuarial context one encounters quite often random variables of the type $\mathbb{S} = \sum_{i=1}^n X_i$ where the terms X_i are not mutually independent, but the multivariate distribution function of the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified because one only knows the marginal distribution functions of the random variables X_i. In such cases, to be able to make decisions it may be helpful to find the dependence structure for the random vector $(X_1, \ldots, X_n)$ producing the least favourable aggregate claims $\mathbb{S}$ with given marginals. Therefore, given the marginal distributions of the terms in a random variable $\mathbb{S} = \sum_{i=1}^n X_i$, we shall look for the joint distribution with a smaller resp. larger sum, in the convex order sense. In short, the sum $\mathbb{S}$ is bounded below and above in convex order ($\preceq_{\text{cx}}$) by sums of comonotonic variables:

$$\mathbb{S}^\ell \preceq_{\text{cx}} \mathbb{S} \preceq_{\text{cx}} \mathbb{S}^u \preceq_{\text{cx}} \mathbb{S}^c,$$

which implies by definition of convex order that

$$E[(\mathbb{S}^\ell - d)_+] \leq E[(\mathbb{S} - d)_+] \leq E[(\mathbb{S}^u - d)_+] \leq E[(\mathbb{S}^c - d)_+]$$

for all d in $\mathbb{R}^+$, while $E[\mathbb{S}^\ell] = E[\mathbb{S}] = E[\mathbb{S}^u] = E[\mathbb{S}^c]$.

2.1. Comonotonic upper bound

As proven in Dhaene et al. (2002), the convex-largest sum of the components of a random vector with given marginals is obtained by the comonotonic sum $\mathbb{S}^c = X_1^c + X_2^c + \cdots + X_n^c$ with

$$\mathbb{S}^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U), \tag{2}$$

where the usual inverse of a distribution function, which is the non-decreasing and left-continuous function $F_X^{-1}(p)$ is defined by

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0,1],$$

with $\inf \emptyset = +\infty$ by convention.

Kaas et al. (2000) have proved that the inverse distribution function of a sum of comonotonic random variables is simply the sum of the inverse distribution functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the cdf $F_{\mathbb{S}^c}(x)$ is uniquely determined by

$$F_{\mathbb{S}^c}^{-1}\left(F_{\mathbb{S}^c}(x)\right) = \sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(x)\right) = x, \qquad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1).$$

Hereafter we restrict ourselves to this case of strictly increasing and continuous marginals.

In the following theorem Dhaene et al. (2002) have proved that the stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms.

Theorem 1. *The stop-loss premiums of the sum $\mathbb{S}^c$ of the components of the comonotonic random vector $(X_1^c, X_2^c, \ldots, X_n^c)$ are given by*

$$E\left[(\mathbb{S}^c - d)_+\right] = \sum_{i=1}^{n} E\left[\left(X_i - F_{X_i}^{-1}\left(F_{\mathbb{S}^c}(d)\right)\right)_+\right], \qquad (F_{\mathbb{S}^c}^{-1}(0) < d < F_{\mathbb{S}^c}^{-1}(1)).$$

If the only information available concerning the multivariate distribution function of the random vector $(X_1, \ldots, X_n)$ are the marginal distribution functions of the X_i, then the distribution function of $\mathbb{S}^c = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \cdots + F_{X_n}^{-1}(U)$ is a prudent choice for approximating the unknown distribution function of $\mathbb{S} = X_1 + \cdots + X_n$. It is a supremum in terms of convex order. It is the best upper bound that can be derived under the given conditions.

2.2. Improved comonotonic upper bound

Let us now assume that we have some additional information available concerning the stochastic nature of $(X_1, \ldots, X_n)$. More precisely, we assume that there exists some random variable Λ with a given distribution function, such that we know the conditional cumulative distribution functions, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. In fact, Kaas et al. (2000) define the improved comonotonic upper bound $\mathbb{S}^u$ as

$$\mathbb{S}^u = F_{X_1|\Lambda}^{-1}(U) + F_{X_2|\Lambda}^{-1}(U) + \cdots + F_{X_n|\Lambda}^{-1}(U),$$

where $F^{-1}_{X_i|\Lambda}(U)$ is the notation for the random variable $f_i(U,\Lambda)$, with the function f_i defined by $f_i(u,\lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$. In order to obtain the distribution function of $\mathbb{S}^u$, observe that given the event $\Lambda = \lambda$, the random variable $\mathbb{S}^u$ is a sum of comonotonic random variables. If the marginal cdfs $F_{X_i|\Lambda=\lambda}$ are strictly increasing and continuous, then $F_{\mathbb{S}^u|\Lambda=\lambda}(x)$ is a solution to

$$\sum_{i=1}^{n} F^{-1}_{X_i \mid \Lambda=\lambda}\left(F_{\mathbb{S}^u \mid \Lambda=\lambda}(x)\right) = x, \qquad x \in \left(F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u \mid \Lambda=\lambda}(1)\right), \tag{3}$$

and the cdf of $\mathbb{S}^u$ then follows from

$$F_{\mathbb{S}^u}(x) = \int_{-\infty}^{+\infty} F_{\mathbb{S}^u|\Lambda=\lambda}(x)\; dF_\Lambda(\lambda).$$

In this case, we also find that for any $d \in \left(F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(0), F^{-1}_{\mathbb{S}^u|\Lambda=\lambda}(1)\right)$:

$$E\left[(\mathbb{S}^u - d)_+ \mid \Lambda = \lambda\right] = \sum_{i=1}^{n} E\left[\left(X_i - F^{-1}_{X_i|\Lambda=\lambda}\left(F_{\mathbb{S}^u|\Lambda=\lambda}(d)\right)\right)_+ \mid \Lambda = \lambda\right], \tag{4}$$

from which the stop-loss premium at retention d of $\mathbb{S}^u$ can be determined by integration with respect to λ.

2.3. Lower bound

Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector with given marginal cdfs $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$. We assume as in the previous section that there exists some random variable Λ with a given distribution function, such that we know the conditional cdfs, given $\Lambda = \lambda$, of the random variables X_i, for all possible values of λ. This random variable Λ, however, should not be the same as in case of the upper bound. We recall from Kaas et al. (2000) that a lower bound, in the sense of convex order, for $\mathbb{S} = X_1 + X_2 + \cdots + X_n$ is

$$\mathbb{S}^\ell = E\left[\mathbb{S} \mid \Lambda\right].$$

This idea can also be found in Rogers and Shi (1995) for the continuous case.
Let us further assume that the random variable Λ is such that all $E\left[X_i \mid \Lambda\right]$ are non-decreasing and continuous functions of Λ and in addition assume that the cdfs of the random variables $E\left[X_i \mid \Lambda\right]$ are strictly increasing and continuous, then the cdf of $\mathbb{S}^\ell$ is also strictly increasing and continuous, and we get for all $x \in \left(F^{-1}_{\mathbb{S}^\ell}(0)\,,\, F^{-1}_{\mathbb{S}^\ell}(1)\right)$,

$$\sum_{i=1}^{n} F^{-1}_{E[X_i|\Lambda]}\left(F_{\mathbb{S}^\ell}(x)\right) = x \quad\Leftrightarrow\quad \sum_{i=1}^{n} E\left[X_i \mid \Lambda = F^{-1}_\Lambda\left(F_{\mathbb{S}^\ell}(x)\right)\right] = x, \tag{5}$$

which unambiguously determines the cdf of the convex order lower bound $\mathbb{S}^\ell$ for $\mathbb{S}$. Applying Theorem 1 and using (5), the stop-loss premiums of $\mathbb{S}^\ell$ can be computed as:

$$E\left[\left(\mathbb{S}^\ell - d\right)_+\right] = \sum_{i=1}^{n} E\left[\left(E\left[X_i \mid \Lambda\right] - E\left[X_i \mid \Lambda = F^{-1}_\Lambda\left(F_{\mathbb{S}^\ell}(d)\right)\right]\right)_+\right], \tag{6}$$

which holds for all retentions $d \in \left(F^{-1}_{\mathbb{S}^\ell}(0)\,,\; F^{-1}_{\mathbb{S}^\ell}(1)\right)$.

So far, we considered the case that all $E[X_i \mid \Lambda]$ are non-decreasing functions of Λ. The case where all $E[X_i \mid \Lambda]$ are non-increasing and continuous functions of Λ also leads to a comonotonic vector $(E[X_1 \mid \Lambda], E[X_2 \mid \Lambda], \ldots, E[X_n \mid \Lambda])$, and can be treated in a similar way.

3. Basket options in a Black & Scholes setting

We now shall concentrate on bounds for the basket and Asian basket option by comonotonicity reasoning and by using the approach of Rogers & Shi which has been generalized by Nielsen and Sandmann (2002) in case of Asian options.

We denote by $S_i(t)$ the price of the i-th asset in the basket at time t. Assume the basket is given by

$$\mathbb{S}(t) = \sum_{i=1}^{n} a_i S_i(t),$$

where a_i are deterministic, positive and constant weights specified by the option contract. We assume under the risk neutral measure Q

$$dS_i(t) = rS_i dt + \sigma_i S_i dW_i(t),$$

where $\{W_i(t), t \geq 0\}$ is a standard Brownian motion associated with the price process of asset i. Further, we assume the different asset prices to be instantaneously correlated according to

$$\text{corr}(dW_i, dW_j) = \rho_{ij} dt. \tag{7}$$

Given the above dynamics, the i-th asset price at time t equals

$$S_i(t) = S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)}.$$

3.1. Bounds based on comonotonicity reasoning

First we note that according to Section 2 it is possible to derive upper and lower bounds for the stop-loss premium $E^Q\left[(\mathbb{S} - d)_+\right]$ where $\mathbb{S}$ is a linear combination of lognormal variables. For the details we refer to Vanmaele et al. (2002). We can rewrite the basket as a sum of lognormal variables

$$\mathbb{S}(t) = \sum_{i=1}^{n} X_i(t) = \sum_{i=1}^{n} \alpha_i(t) e^{Y_i(t)}, \tag{8}$$

where $\alpha_i(t) = a_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)t}$ and $Y_i(t) = \sigma_i W_i(t) \sim N(0, \sigma_i^2 t)$ and thus $X_i(t)$ is lognormally distributed: $X_i(t) \sim LN(\ln(a_i S_i(0)) + (r - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)$.
In this case the stop-loss premium with some retention d_i, namely $E^Q\left[(X_i - d_i)_+\right]$, is well-known since $\ln(X_i(t)) \sim N\left(\mu_{i(t)}, \sigma^2_{Y_i(t)}\right)$ with $\mu_i(t) = \ln(\alpha_i(t))$ and $\sigma_{Y_i(t)} = \sigma_i \sqrt{t}$, and equals for $d_i > 0$

$$E^Q[(X_i(t) - d_i)_+] = e^{\mu_i(t) + \frac{\sigma^2_{Y_i(t)}}{2}} \Phi(d_{i,1}(t)) - d_i \Phi(d_{i,2}(t)), \tag{9}$$

where $d_{i,1}$ and $d_{i,2}$ are determined by

$$d_{i,1}(t) = \frac{\mu_i(t) + \sigma^2_{Y_i(t)} - \ln(d_i)}{\sigma_{Y_i(t)}}, \quad d_{i,2}(t) = d_{i,1}(t) - \sigma_{Y_i(t)}, \tag{10}$$

and where Φ is the cdf of the $N(0,1)$ distribution.
The case $d_i < 0$ is trivial.

In what follows we only consider the basket at maturity date T and for the sake of notational simplicity, we shall not longer denote explicitly the dependence on T in X_i, α_i and Y_i.

3.1.1. *Comonotonic upper bound.* By the lognormality of the components $X_i = a_i S_i$ in the sum $\mathbb{S}$ (8), the inverse cdfs $F_{X_i}^{-1}$ can easily be derived, leading in (2) to

$$\mathbb{S}^c = \sum_{i=1}^n \alpha_i e^{\frac{1}{2}\sigma_{Y_i}^2 \Phi^{-1}(U)},$$

for any random variable U which is uniformly distributed on the unit interval. Combining Theorem 1 and (9)-(10), and substituting α_i and the standard deviation of Y_i, we obtain the following comonotonic upper bound for any $K > 0$:

$$BC(n,K,T) \leq \sum_{i=1}^n a_i S_i(0)\Phi\left[\sigma_i\sqrt{T} - \Phi^{-1}\left(F_{\mathbb{S}^c}(K)\right)\right] - e^{-rT}K\left(1 - F_{\mathbb{S}^c}(K)\right),$$

where $F_{\mathbb{S}^c}(K)$ follows from

$$\sum_{i=1}^n a_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)T+\ \sigma_i\sqrt{T}\,\Phi^{-1}(F_{\mathbb{S}^c}(K))} = K.$$

Similarly as for the Asian options in Simon, Goovaerts and Dhaene (2000), we can rewrite this upper bound as a combination of Black & Scholes prices. In fact, noting that by the lognormality of S_i

$$F_{a_iS_i}^{-1}(p) = a_i F_{S_i}^{-1}(p) \quad \text{for all } p \in [0,1],$$

it follows from comonotonicity results that it is the smallest linear/weighted combination of Black & Scholes European call prices dominating the basket option price:

$$\begin{aligned} BC(n,K,T) &\leq e^{-rT}\sum_{i=1}^n a_i E^Q\left[(S_i - F_{S_i}^{-1}(F_{\mathbb{S}^c}(K)))_+\right] \\ &= \sum_{i=1}^n a_i\left(S_i(0)\Phi(d_{i1}) - e^{-rT}K_i\Phi(d_{i2})\right), \end{aligned}$$

with

$$K_i = F_{S_i}^{-1}(F_{\mathbb{S}^c}(nK)) = S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)T+\sigma_i\sqrt{T}\,\Phi^{-1}(F_{\mathbb{S}^c}(K))}$$

$$d_{i1} = \frac{\ln\left(\frac{S_i(0)}{K_i}\right) + (r + \frac{\sigma_i^2}{2})T}{\sigma_i\sqrt{T}}, \quad d_{i2} = d_{i1} - \sigma_i\sqrt{T}.$$

Remark that this comonotonic upper bound is independent of ρ_{ij} (7), which implies that even when in practice these correlations are not known we have an upper bound for the price of the basket option. It is intuitively clear that this upper bound can be improved by taking these correlations into account.

3.1.2. *Improved comonotonic upper bound.* Following Section 2.2, we improve the comonotonic upper bound $\mathbb{S}^c$ (in convex order) for $\mathbb{S}$ (8) by conditioning on some normally distributed random variable Λ such that $Y_i \mid \Lambda$ is also normally distributed for all i with parameters $\mu(i) = r_i \frac{\sigma_{Y_i}}{\sigma_\Lambda}\left(\lambda - E^Q[\Lambda]\right)$ and $\sigma^2(i) = \left(1 - r_i^2\right)\sigma_{Y_i}^2$:

$$\mathbb{S}^u = \sum_{i=1}^n \alpha_i e^{r_i \sigma_{Y_i}\Phi^{-1}(V) + \sqrt{1-r_i^2}\sigma_{Y_i}\Phi^{-1}(U)},$$

where U and $V = \Phi\left(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}\right)$ are mutually independent uniform(0,1) random variables, Φ is the cdf of the $N(0,1)$ distribution and r_i is defined by

$$r_i = r\left(Y_i, \Lambda\right) = \frac{\mathrm{cov}\left(Y_i, \Lambda\right)}{\sigma_{Y_i}\sigma_\Lambda}.$$

Combining (4) with (9)-(10) and substituting α_i and the standard deviation of Y_i, we construct the improved comonotonic upper bound for the basket price $BC(n, K, T)$:

$$e^{-rT} E^Q\left[(\mathbb{S}^u - K)_+\right] = \sum_{i=1}^n a_i S_i(0) e^{-\frac{1}{2}\sigma_i^2 r_i^2 T} \times \tag{11}$$

$$\times \int_0^1 e^{r_i \sigma_i \sqrt{T}\Phi^{-1}(v)} \Phi\left(\sqrt{1-r_i^2}\sigma_i\sqrt{T} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(K)\right)\right) dv - e^{-rT} K \left(1 - F_{\mathbb{S}^u}(K)\right),$$

where the conditional distribution $F_{\mathbb{S}^u|V=v}(K)$ is, according to (3), determined by

$$\sum_{i=1}^n a_i S_i(0) \exp\left[(r - \frac{\sigma_i^2}{2})T + r_i\sigma_i\sqrt{T}\Phi^{-1}(v) + \sqrt{1-r_i^2}\sigma_i\sqrt{T}\Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(K)\right)\right] = K,$$

and integration with respect to v gives:

$$F_{\mathbb{S}^u}(K) = \int_0^1 F_{\mathbb{S}^u|V=v}(K) dv.$$

We now discuss the choice of the conditioning variable Λ which should not only be normally distributed but also such that (Y_i, Λ) for all i are bivariate normally distributed. Hence, we define Λ by

$$\Lambda = \sum_{i=1}^n \beta_i \sigma_i W_i(T) \tag{12}$$

with β_i some real numbers. The correlation between Y_i and Λ is given by

$$r_i = \frac{\mathrm{cov}\left(\sigma_i W_i(T), \Lambda\right)}{\sqrt{T}\sigma_i\sigma_\Lambda} = \frac{\sum_{j=1}^n \beta_j \rho_{ij}\sigma_j}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \beta_i\beta_j\rho_{ij}\sigma_i\sigma_j}}. \tag{13}$$

In this paper we consider the following types of conditioning variable Λ.

- As a first conditioning variable we take a linear transformation of a first order approximation of $\mathbb{S}$ (denoted by $FA1$):

$$FA1 = \sum_{i=1}^{n} e^{(r-\frac{\sigma_i^2}{2})T} a_i S_i(0)\sigma_i W_i(T), \tag{14}$$

and the correlation coefficients then read

$$r_i = \frac{\sum_{j=1}^{n} a_j S_j(0) e^{(r-\frac{\sigma_j^2}{2})T}\rho_{ij}\sigma_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i S_i(0) e^{(r-\frac{\sigma_i^2}{2})T} a_j S_j(0) e^{(r-\frac{\sigma_j^2}{2})T}\rho_{ij}\sigma_i\sigma_j}}. \tag{15}$$

- As a second conditioning variable (denoted by $FA2$), we consider

$$FA2 = \sum_{i=1}^{n} a_i S_i(0)\sigma_i W_i(T). \tag{16}$$

In this case, the correlation between Y_i and Λ is easily found to be

$$r_i = \frac{\sum_{j=1}^{n} a_j S_j(0)\rho_{ij}\sigma_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i S_i(0) a_j S_j(0)\rho_{ij}\sigma_i\sigma_j}}. \tag{17}$$

Note that $FA2$ is also a first order approximation of $\mathbb{S}$ and in fact of $FA1$.

- As a third conditioning variable (denoted by GA), we look at the standardized logarithm of the geometric average $\mathbb{G}$ which is defined by

$$\mathbb{G} = \prod_{i=1}^{n} S_i(T)^{a_i} = \prod_{i=1}^{n}\left(S_i(0)e^{(r-\frac{\sigma_i^2}{2})T}\right)^{a_i}. \tag{18}$$

Indeed, we can consider

$$GA = \frac{\ln\mathbb{G} - E^Q[\ln\mathbb{G}]}{\sqrt{\mathrm{var}[\ln\mathbb{G}]}} = \frac{\sum_{i=1}^{n} a_i\sigma_i W_i(T)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\sigma_i\sigma_j\rho_{ij}T}}, \tag{19}$$

since

$$E^Q[\ln\mathbb{G}] = \sum_{i=1}^{n} a_i\left(\ln(S_i(0)) + (r-\frac{\sigma_i^2}{2})T\right) \tag{20}$$

$$\mathrm{var}[\ln\mathbb{G}] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\sigma_i\sigma_j\rho_{ij}T. \tag{21}$$

The correlation coefficients in this case are given by

$$r_i = \frac{\mathrm{cov}\left(\sigma_i W_i(T), \Lambda\right)}{\sqrt{T}\sigma_i\sigma_\Lambda} = \frac{\sum_{j=1}^{n} a_j\sigma_j\rho_{ij}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\rho_{ij}\sigma_i\sigma_j}}. \tag{22}$$

The improved comonotonic upper bound can be computed separately for each of the choices (14), (16) and (19).

Similarly to the comonotonic upper bound, we can rewrite the improved comonotonic upper bound as a combination of Black & Scholes prices. For this purpose, given $\Lambda = \lambda$ or equivalently given $V = v$, we introduce some artificial underlying assets $\tilde{S}_{i,v}$ having volatilities $\tilde{\sigma}_{i,v} = \sigma_i \sqrt{1 - r_i^2}$ and with initial value

$$\tilde{S}_{i,v}(0) = S_i(0) e^{-\frac{1}{2}\sigma_i^2 r_i^2 T + r_i \sigma_i \sqrt{T}\, \Phi^{-1}(v)}.$$

We also consider new exercise prices:

$$\tilde{K}_{i,v} = S_i(0) e^{(r - \frac{\sigma_i^2}{2})T + r_i \sigma_i \sqrt{T} \Phi^{-1}(v) + \sqrt{1 - r_i^2}\sigma_i \sqrt{T}\, \Phi^{-1}(F_{\mathbb{S}^u|V=v}(K))}.$$

3.1.3. *Lower bound.* Following Section 2.3 we condition on some random variable Λ in order to derive a lower bound. For our purpose, we take (12) as the conditioning variable and in particular, we consider the three choices *FA*1, *FA*2 and *GA*, mentioned above. Noting that $Y_i \mid \Lambda$ is normally distributed with parameters $\mu(i)$ and $\sigma^2(i)$ as in Section 3.1.2, we easily arrive at

$$\mathbb{S}^\ell \equiv \sum_{i=1}^n E^Q \left[S_i(T) \mid \Lambda \right] = \sum_{i=1}^n a_i S_i(0) e^{(r - \frac{\sigma_i^2}{2} r_i^2)T + \sigma_i r_i \sqrt{T} \Phi^{-1}(V)}, \tag{23}$$

where the random variable $V = \Phi\left(\frac{\Lambda - E^Q[\Lambda]}{\sigma_\Lambda}\right)$ is uniformly distributed on the unit interval. This sum is a sum of n comonotonic risks under the assumption that for all i the correlations r_i are positive. We shall come back to this issue later on.
Invoking (6) and (9)-(10), and substituting α_i and the standard deviation of Y_i, we find the following lower bound for the price of the basket call option:

$$BC(n,K,T) \geq \sum_{i=1}^n a_i S_i(0)\; \Phi\left[\sigma_i \sqrt{T} r_i - \Phi^{-1}\left(F_{\mathbb{S}^\ell}(K)\right)\right] - e^{-rT}\; K\left(1 - F_{\mathbb{S}^\ell}(K)\right) \tag{24}$$

which holds for any $K > 0$ and where $F_{\mathbb{S}^\ell}(K)$, according to (5), solves

$$\sum_{i=1}^n a_i S_i(0) e^{(r - \frac{1}{2} r_i^2 \sigma_i^2)T +\; r_i \sigma_i \sqrt{T}\, \Phi^{-1}\left(F_{\mathbb{S}^\ell}(K)\right)} = K. \tag{25}$$

Similarly as for the upper bounds, the lower bound (24)-(25) can be formulated as an average of Black & Scholes formulae with new underlying assets and new exercise prices. The new assets $\tilde{S}_i$ are with $\tilde{S}_i(0) = S_i(0)$ and with new volatilities $\tilde{\sigma}_i = \sigma_i r_i$ for $i = 1, \ldots, n$. The new exercise prices $\tilde{K}_i$, $i = 1, \ldots, n$, are given by

$$\tilde{K}_i = \tilde{S}_i(0) e^{(r - \frac{\tilde{\sigma}_i^2}{2})T + \tilde{\sigma}_i \sqrt{T} \Phi^{-1}\left(F_{\mathbb{S}^\ell}(K)\right)}.$$

Indeed,

$$BC(n,K,T) \geq \sum_{i=1}^n a_i \left[\tilde{S}_i(0) \Phi\left(d_{1i}\right) - e^{-rT} \tilde{K}_i\; \Phi\left(d_{2i}\right)\right] \tag{26}$$

with

$$d_{1i} = \frac{\ln\left(\frac{\tilde{S}_i(0)}{\tilde{K}_i}\right) + (r + \frac{\tilde{\sigma}_i^2}{2})T}{\tilde{\sigma}_i\sqrt{T}} \quad \text{and} \quad d_{2i} = d_{1i} - \tilde{\sigma}_i\sqrt{T}, \quad \text{for } i = 1, \ldots, n.$$

Beißer (2001) has obtained the same result by using other arguments. Further remark that in case r_i equals one, the lower bound coincides with the comonotonic upper bound and we obtain the exact price. In practice we did not find up to now a conditioning variable Λ such that $r_i = 1$ for all i. But we do have that for the conditioning variables (14), (16) and (19) the lower bound is quite good. Beißer (2001) chooses along intuitive arguments the numerator of the standardized logarithm of the geometric average (19). This is indeed a good choice since the geometric average and arithmetic average are based on the same information. In this case, the correlation coefficients in the formulae for the lower bound are given by (22). Note however that these correlation coefficients are independent of the initial value of the assets in the basket which can lead to a lower quality of the lower bound when the assets in the basket have different initial values. Beißer (2001) therefore considers also the conditioning variable Λ given by (16) with correlations r_i (17), which depend on the weights, the initial values and the volatilities of the assets in the basket.
It is easily seen that the lower bound will coincide for the three different choices of Λ when the initial values as well as the volatilities are equal for the different assets. When only the volatilities σ_i are equal for all i then the correlation coefficients (15) and (17) coincide and hence also the corresponding lower bounds. Similarly, when only the initial values are equal the correlation coefficients (17) and (22) lead to the same lower bound.

Next we go deeper into the assumption of positiveness for the correlation coefficients r_i (13). This condition is needed for $\mathbb{S}^\ell$ (23) to be a comonotonic sum.
When the correlations ρ_{ij} (7) are positive for all i and j then it suffices to take all coefficients β_i also with a positive sign in order to satisfy the assumption. However when a ρ_{ij} is negative a general discussion is much more involved. Therefore, we first look at the special case when $n = 2$ and $\rho_{12} = \rho_{21} \overset{not}{=} \rho \leq 0$. The conditions $r_1, r_2 \geq 0$ are equivalent to

$$\begin{cases} \beta_1\sigma_1 - \beta_2\sigma_2|\rho| \geq 0 \\ \beta_2\sigma_2 - \beta_1\sigma_1|\rho| \geq 0 \end{cases} \quad \Leftrightarrow \quad \beta_2\sigma_2|\rho| \leq \beta_1\sigma_1 \leq \beta_2\sigma_2\frac{1}{|\rho|}, \tag{27}$$

and imply that β_1 and β_2 should have the same sign and differ from zero. For simplicity assume that β_1 and β_2 are both strictly positive, then the condition (27) can be rewritten as

$$|\rho| \leq \frac{\beta_1\sigma_1}{\beta_2\sigma_2} \leq \frac{1}{|\rho|}. \tag{28}$$

Note that since $|\rho| \leq 1$, the second inequality is trivially fulfilled when $\beta_1\sigma_1 \leq \beta_2\sigma_2$ while in the case $\beta_1\sigma_1 \geq \beta_2\sigma_2$ the first inequality is trivial. Hence only one of these inequalities has to be checked. Beißer (2001) made a similar reasoning but only for the particular correlation coefficients (17).
When ρ is negative, it can happen that for none of the three choices for Λ, namely (14), (16) and (19), relation (28) is satisfied. However, since we derived a lower bound for any Λ given by (12) we are not restricted to the three choices. Indeed, it is always

possible to find a β_1 and β_2 since the interval $[|\rho|\frac{\sigma_2}{\sigma_1}, \frac{1}{|\rho|}\frac{\sigma_2}{\sigma_1}]$ is non-empty.
In fact, one might search for the β_1 and β_2 which leads to an optimal lower bound. When we write r_i, $i = 1, 2$ in function of $x = \frac{\beta_2\sigma_2}{\beta_1\sigma_1}$, we find that $r_1 = \frac{1+x\rho}{\sqrt{1+x^2+2x\rho}}$ and $r_2 = \frac{\rho+x}{\sqrt{1+x^2+2x\rho}}$. If one assumes that $r_2 \neq 1$ and if we rewrite the equation defining r_2, we find the relation $r_1 - r_2\rho = \sqrt{(1-\rho^2)(1-r_2^2)}$. As a consequence, the optimal lower bound becomes the solution to the optimization program

$$\max_{r_1, r_2} LB(r_1, r_2) = \sum_{i=1}^{2} a_i[\tilde{S}_i(0)\Phi(d_{1i}) - e^{-rT}\tilde{K}_i\Phi(d_{1i})] \tag{29}$$

$$\text{such that } 0 \leq r_1 \leq 1, 0 \leq r_2 < 1$$

$$\sum_{i=1}^{2} a_i\tilde{K}_i = K$$

$$r_1 - r_2\rho = \sqrt{1-\rho^2}\sqrt{1-r_2^2},$$

where we used the notation (25)-(26). Solving this problem by Lagrange optimization leads to a conclusion of three cases:

1. If $r_2 = 0$ and $r_1 = \sqrt{1-\rho^2}$, which is only possible if $\rho < 0$, the lower bound is maximized under the above conditions if

$$a_2\tilde{K}_2\sigma_2 + \rho\sigma_1(K - a_2\tilde{K}_2) \leq 0 \quad \text{with} \quad \tilde{K}_2 = S_2(0)e^{rT}.$$

2. If r_1 and r_2 are strictly between 0 and 1, r_1 and r_2 are solutions to the equations:

$$\begin{cases} e^{-rT}\varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(K)))a_1\tilde{K}_1\sigma_1\sqrt{T} - \lambda_2 = 0 \\ e^{-rT}\varphi(\Phi^{-1}(F_{\mathbb{S}^\ell}(K)))a_2\tilde{K}_2\sigma_2\sqrt{T} - \lambda_2\left(-\rho + r_2\sqrt{\frac{1-\rho^2}{1-r_2^2}}\right) = 0 \\ a_1\tilde{K}_1 + a_2\tilde{K}_2 = K \\ r_1 - r_2\rho = \sqrt{1-\rho^2}\sqrt{1-r_2^2}, \end{cases}$$

which are four non-linear equations in four unknowns $r_1, r_2, \lambda_2, \Phi^{-1}(F_{\mathbb{S}^\ell}(K))$ and where φ is the density function of the $N(0,1)$ distribution.

3. If $r_1 = 0$ and $r_2 = \sqrt{1-\rho^2}$, which is only possible if $\rho < 0$, the lower bound is maximized under the above conditions if

$$a_1\tilde{K}_1\sigma_1 + \rho\sigma_2(K - a_1\tilde{K}_1) \leq 0 \quad \text{with} \quad \tilde{K}_1 = S_1(0)e^{rT}.$$

We now turn to the general case for $n \geq 3$ and at least one correlation ρ_{ij}, (7), is strictly negative. As a conclusion of the following statement we see that a lower bound can be computed also in a general case. However, the optimization program will be much more involved when there are more than two assets in the basket.

Theorem 2. *There always exist coefficients $\beta_i \in \mathbb{R}$, $i = 1, \ldots, n$, in* (12) *such that all correlations r_i, $i = 1, \ldots, n$,* (13) *are positive.*

Proof. Denoting $\mathbf{A}$ for the correlation matrix $(\rho_{ij})_{1\leq i,j\leq n}$ and putting $\boldsymbol{\beta}^T = (\beta_1\sigma_1, \ldots, \beta_n\sigma_n)$, the conditions $r_i \geq 0$, $i = 1, \ldots, n$ are equivalent to $\mathbf{A}\boldsymbol{\beta} \geq \mathbf{0}$.
As all variance-covariance matrices, this matrix $\mathbf{A}$ is symmetric and positive semi-definite. Moreover it is non-singular and positive definite since we assume that the market is complete.
By a reasoning ex absurdo we show that at least one of the coefficients β_i is strictly positive: Assume that all β_i are negative then from $\mathbf{A}\boldsymbol{\beta} \geq \mathbf{0}$ it follows that $\boldsymbol{\beta}^T\mathbf{A}\boldsymbol{\beta} \leq \mathbf{0}$ which is a contradiction to the positive definiteness of $\mathbf{A}$.
Finally, using the link between the primal and the dual of a linear programming problem, the assertion can then be proved. □

Note that one could apply the above optimization procedure to the improved comonotonic upper bound, however we do not expect that the improvement resulting from the optimization would lead to the best upper bounds. For a more detailed discussion we refer to Section 5.

3.2. Bounds based on the Rogers & Shi approach

Following the ideas of Rogers and Shi (1995), we derive an upper bound based on the lower bound. Indeed, we obtain an error bound

$$0 \leq E^Q\left[E^Q\left[(\mathbb{S}-K)_+ \mid \Lambda\right] - (\mathbb{S}^\ell - K)_+\right] \leq \frac{1}{2}E^Q\left[\sqrt{\mathrm{var}(\mathbb{S}\mid\Lambda)}\right]. \tag{30}$$

Consequently, we find as upper bound for the arithmetic basket option

$$BC(n,K,T) \leq e^{-rT}\left\{E^Q\left[(\mathbb{S}^\ell - K)_+\right] + \frac{1}{2}E^Q\left[\sqrt{\mathrm{var}(\mathbb{S}\mid\Lambda)}\right]\right\}. \tag{31}$$

Using properties of lognormal distributed variables, the second term on the right hand side can be written out explicitly, giving some lenghty, analytical, computable expression:

$$E^Q\left[\sqrt{\mathrm{var}(\mathbb{S}\mid\Lambda)}\right] = E^Q\left[\left(\sum_{i=1}^n\sum_{j=1}^n E^Q\left[a_ia_jS_i(T)S_j(T)\mid\Lambda\right] - \left(\mathbb{S}^\ell\right)^2\right)^{1/2}\right], \tag{32}$$

where $\mathbb{S}^\ell$ was defined in (23) and where the first term in the right hand side equals

$$\sum_{i=1}^n\sum_{j=1}^n a_ia_jS_i(0)S_j(0)e^{(2r-\frac{\sigma_i^2+\sigma_j^2}{2})T + r_{ij}\sigma_{ij}\sqrt{T}\Phi^{-1}(U)+\frac{1}{2}(1-r_{ij}^2)T\sigma_{ij}^2}, \tag{33}$$

with $\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 + 2\sigma_i\sigma_j\rho_{ij}$ and $r_{ij} = \frac{\sigma_i}{\sigma_{ij}}r_i + \frac{\sigma_j}{\sigma_{ij}}r_j$, and where U is uniformly distributed on the interval $(0,1)$.

This upper bound holds for any choice of the coefficients β_i in the expression of Λ (12) as long as the correlations r_i are positive. This allows us to take the minimum over several upper bounds. Note also that the error bound (32) is independent of the strike K.

For Asian option pricing, Nielsen and Sandmann (2002) were able to improve the Rogers & Shi methodology. We succesfully adapt that approach to the setting of basket

options.
For any $d \in \mathbb{R}$ such that $\Lambda \geq d$ implies $\mathbb{S} \geq K$, it follows that

$$E^Q[(\mathbb{S}-K)_+ \mid \Lambda] = E^Q[\mathbb{S}-K \mid \Lambda] = \left(\mathbb{S}^\ell - K\right)_+$$

and, denoting by $f_\Lambda(\cdot)$ the normal density function for Λ, that the error bound (30) can be replaced by

$$\begin{aligned} 0 &\leq E^Q\left[E^Q[(\mathbb{S}-K)_+ \mid \Lambda] - \left(\mathbb{S}^\ell - K\right)_+\right] \\ &= \int_{-\infty}^{d} \left(E^Q[(\mathbb{S}-K)_+ \mid \Lambda=\lambda] - \left(E^Q\left[\mathbb{S} \mid \Lambda=\lambda\right] - K\right)_+\right) f_\Lambda(\lambda) d\lambda \\ &\leq \frac{1}{2}\int_{-\infty}^{d} \left(\operatorname{var}\left(\mathbb{S} \mid \Lambda=\lambda\right)\right)^{\frac{1}{2}} f_\Lambda(\lambda) d\lambda \\ &\leq \frac{1}{2}\left(E^Q\left[\operatorname{var}\left(\mathbb{S} \mid \Lambda\right) 1_{\{\Lambda<d\}}\right]\right)^{\frac{1}{2}} \left(E^Q\left[1_{\{\Lambda<d\}}\right]\right)^{\frac{1}{2}}, \end{aligned} \tag{34}$$

where Hölder's inequality has been applied in the last inequality and where $1_{\{\Lambda<d\}}$ is the indicator function.
The upper bound (31) corresponds to the limiting case where d equals infinity.

We can determine d for the three different Λ's (14), (16) and (18), such that $\Lambda \geq d_\Lambda$ implies that $\mathbb{S} \geq K$. Bounding the exponential function e^x below by its first order approximation $1+x$ with $x = \sigma_i W_i(T)$, respectively $x = \left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i W_i(T)$, the integration bound corresponding to $\Lambda = FA1$ given by (14), respectively to $\Lambda = FA2$ given by (16), is found to be

$$d_{FA1} = K - \sum_{i=1}^{n} a_i S_i(0) e^{(r-\frac{\sigma_i^2}{2})T}, \tag{35}$$

$$\text{respectively,} \quad d_{FA2} = K - \sum_{i=1}^{n} a_i S_i(0)(1 + (r - \frac{\sigma_i^2}{2})T). \tag{36}$$

When Λ is the standardized logarithm of the geometric average (GA), see (19), we use the relationship $\mathbb{S} \geq \mathbb{G} \geq K$ and (20)–(21) in order to arrive at

$$d_{GA} = \frac{\ln(K) - \sum_{i=1}^{n} a_i \left(\ln(S_i(0)) + (r - \frac{\sigma_i^2}{2})T\right)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j \rho_{ij} T}}. \tag{37}$$

We remark that for the optimized choice of Λ such d cannot be determined.

Note that in contrast to (32), the error bound (34) now depends on K through d. Therefore we expect that the upper bound containing the error bound (34) will be sharper than the corresponding upper bound with error bound (32). However we should draw the attention to the fact that in the error bound (34) an additional error is introduced through Hölder's inequality which can be larger than the improvement by the use of the integration bound d.

Now we shall derive an easily computable expression for (34).
The second expectation term in the product (34) equals $F_\Lambda(d)$, where $F_\Lambda(\cdot)$ denotes

the normal cumulative distribution function of Λ. The first expectation term in the product (34) can be expressed as

$$E^Q\left[\mathrm{var}\left(\mathbb{S}|\Lambda\right)1_{\{\Lambda<d\}}\right] = E^Q\left[E^Q[\mathbb{S}^2|\Lambda]1_{\{\Lambda<d\}}\right] - E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d\}}\right]. \quad (38)$$

The second term of the right-hand side of (38) can according to (23) be rewritten as

$$E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d\}}\right] = \int_{-\infty}^{d} (E^Q[\mathbb{S}|\Lambda=\lambda])^2 f_\Lambda(\lambda)d\lambda$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0)S_j(0)e^{(2r-\frac{\sigma_i^2 r_i^2+\sigma_j^2 r_j^2}{2})T}\int_{-\infty}^{d} e^{(\sigma_i r_i+\sigma_j r_j)\sqrt{T}\Phi^{-1}(v)} f_\Lambda(\lambda)d\lambda, \quad (39)$$

where we recall that $\Phi^{-1}(v) = \frac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. Applying the equality

$$\int_{-\infty}^{d} e^{b\Phi^{-1}(v)} f_\Lambda(\lambda)d\lambda = e^{\frac{b^2}{2}}\Phi\left(d^*-b\right), \quad d^* = \frac{d-E^Q[\Lambda]}{\sigma_\Lambda}, \quad (40)$$

with $b = (\sigma_i r_i + \sigma_j r_j)\sqrt{T}$ we can express $E^Q\left[(E^Q[\mathbb{S}|\Lambda])^2 1_{\{\Lambda<d\}}\right]$ as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0)S_j(0)e^{(2r+\sigma_i\sigma_j r_i r_j)T}\Phi\left(d^* - (r_i\sigma_i+\sigma_j r_j)\sqrt{T}\right). \quad (41)$$

To transform the first term of the right-hand side of (38) we invoke (33)and apply (40) with $b = r_{ij}\sigma_{ij}\sqrt{T} = (r_i\sigma_i + \sigma_j r_j)\sqrt{T}$:

$$E^Q\left[E^Q[\mathbb{S}^2 \mid \Lambda]1_{\{\Lambda<d\}}\right]$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0)S_j(0)e^{(2r-\frac{\sigma_i^2+\sigma_j^2}{2})T+\frac{1}{2}\left(1-r_{ij}^2\right)\sigma_{ij}^2 T}\int_{-\infty}^{d} e^{r_{ij}\sigma_{ij}\sqrt{T}\Phi^{-1}(v)} f_\Lambda(\lambda)d\lambda$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0)S_j(0)e^{(2r+\sigma_i\sigma_j\rho_{ij})T}\Phi\left(d^* - (r_i\sigma_i+\sigma_j r_j)\sqrt{T}\right). \quad (42)$$

Combining (41) and (42) into (38), and then substituting $F_\Lambda(d)$ and (38) into (34) we get the following expression for the error bound, shortly denoted by $\varepsilon(d)$

$$\varepsilon(d) = \frac{1}{2}\left\{F_\Lambda(d)\right\}^{\frac{1}{2}}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0)S_j(0)e^{(2r+\sigma_i\sigma_j r_i r_j)T}\Phi\left(d^* - (r_i\sigma_i+\sigma_j r_j)\sqrt{T}\right)\times\right.$$
$$\left.\times\left(e^{\sigma_i\sigma_j(\rho_{ij}-r_i r_j)T}-1\right)\right\}^{1/2}. \quad (43)$$

3.3. Partially exact/comonotonic upper bound

We combine the technique for obtaining an improved comonotonic upper bound by conditioning on some normally distributed random variable Λ and the idea of Nielsen

and Sandmann (2002) described in the previous subsection, in order to develop another upper bound.
This so-called partially exact/comonotonic upper bound consists of an exact part of the option price and some improved comonotonic upper bound for the remaining part. This idea of decomposing the calculations goes at least back to Curran (1994).

For any normally distributed random variable Λ, with cdf $F_\Lambda(\cdot)$, for which there exists a d such that $\Lambda \geq d$ implies $\mathbb{S} \geq K$ and for which $Y_i \mid \Lambda$ is also normally distributed for all i, the second term in the equality

$$e^{-rT} E^Q[(\mathbb{S}-K)_+] = e^{-rT} E^Q[E^Q[(\mathbb{S}-K)_+ \mid \Lambda]] \tag{44}$$
$$= e^{-rT}\left\{\int_{-\infty}^{d} E^Q[(\mathbb{S}-K)_+ \mid \Lambda=\lambda] dF_\Lambda(\lambda) + \int_{d}^{+\infty} E^Q[\mathbb{S}-K \mid \Lambda=\lambda] dF_\Lambda(\lambda)\right\}$$

can be written in closed-form along similar lines as (39)-(41):

$$e^{-rT}\int_{d}^{+\infty} E^Q[\mathbb{S} \mid \Lambda=\lambda] f_\Lambda(\lambda)d\lambda - e^{-rT}K(1-F_\Lambda(d))$$
$$= e^{-rT}\sum_{i=1}^{n} a_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2 r_i^2)T}\int_{d}^{+\infty} e^{r_i\sigma_i\sqrt{T}\,\Phi^{-1}(v)} f_\Lambda(\lambda)d\lambda - e^{-rT}K(1-\Phi(d^*))$$
$$= \sum_{i=1}^{n} a_i S_i(0)\Phi(r_i\sigma_i\sqrt{T}-d^*) - e^{-rT}K\Phi(-d^*), \tag{45}$$

where $d^* = \dfrac{d - E^Q[\Lambda]}{\sigma_\Lambda}$ and $v = \dfrac{\lambda - E^Q[\Lambda]}{\sigma_\Lambda}$.
In the first term of (44) we replace $\mathbb{S}$ by $\mathbb{S}^u$ in order to obtain an upper bound and apply (11) but now with an integral from zero to $\Phi(d^*)$:

$$e^{-rT}\int_{-\infty}^{d} E^Q[(\mathbb{S}-K)_+ \mid \Lambda=\lambda] f_\Lambda(\lambda)d\lambda$$
$$\leq e^{-rT}\int_{-\infty}^{d} E^Q[(\mathbb{S}^u-K)_+ \mid \Lambda=\lambda] f_\Lambda(\lambda)d\lambda = e^{-rT}\int_{0}^{\Phi(d^*)} E^Q[(\mathbb{S}^u-K)_+ \mid V=v]\,dv$$
$$= \sum_{i=1}^{n} a_i S_i(0) e^{-\frac{1}{2}\sigma_i^2 r_i^2 T}\int_{0}^{\Phi(d^*)} e^{r_i\,\sigma_i\sqrt{T}\,\Phi^{-1}(v)}\Phi\left(\sqrt{1-r_i^2}\,\sigma_i\sqrt{T} - \Phi^{-1}\left(F_{\mathbb{S}^u|V=v}(K)\right)\right)dv$$
$$- e^{-rT}K\left(\Phi(d^*) - \int_{0}^{\Phi(d^*)} F_{\mathbb{S}^u|V=v}(K)\,dv\right). \tag{46}$$

For the random variables Λ given by (14), (16) and (19) we derived a d, see (35), (36) and (37), and thus we can compute the new upper bound.

4. General remarks

In this section we summarize some general remarks:

- The price of the basket *put* option with exercise date T, n underlying assets and fixed exercise price K, given by $BP(n,K,T) = e^{-rT}E^Q\left[(K-\mathbb{S}(T))_+\right]$ satisfies the put-call parity at the present: $BC(n, K, T) - BP(n, K, T) = \mathbb{S}(0) - e^{-rT}K$. Hence, we can derive bounds for the basket put option from the bounds for the call. These bounds for the put option coincide with the bounds that are obtained by applying the theory of comonotonic bounds or the Rogers and Shi approach directly to basket put options. This stems from the fact that the put-call parity also holds for these bounds.

- The case of a continuous dividend yield q_i can easily be dealt with by replacing the interest rate r by $r - q_i$.
- For $n = 1$ there is only one asset in the basket and hence the comonotonic sums $\mathbb{S}^c$, $\mathbb{S}^u$ and $\mathbb{S}^\ell$ all three coincide with the sum $\mathbb{S}$ which consists of only one term: this asset. In this case, the comonotonic upper and lower bounds, including the partially exact/comonotonic upper bound, reduce to the well-known Black & Scholes price for an option on a single asset. This is also true for the bounds based on the Rogers & Shi approach since the error bound is zero.
- As for the Asian options (see Vanmaele et al. (2002)), we can easily derive the hedging Greeks for the upper and lower bounds of a basket option since we found analytical expressions for these bounds. Moreover the expressions are in terms of Black & Scholes prices.

5. Numerical illustration

In this section we give a number of numerical examples on basket options in the Black & Scholes setting.
The first set of input data was taken from Arts (1999). Note that we consider here the forward-moneyness, which is defined as the ratio of the forward price of the basket and the exercise price K. The input parameters correspond to a two-dimensional basket. We first consider equal weights and afterwards, unequal weights. The spot prices are first assumed to be equal to 100 units, and then allowed to vary. The risk-free interest rate is fixed at 5% and we assume no dividends. Moneyness ranges from 10% in-the-money to 10% out-of-the-money. For the time to maturity T two cases are considered ($T = 1, 3$ years). For the correlation (7), two values are considered, representing low and high correlation respectively. We consider equal volatilities (high and low) for both individual assets in the basket.
Concerning the upper bounds, we present only the results that lead to the best upper bound together with the corresponding type of the bound. That is, the upper bound given in the Tables 1 − 3 is the bound which satisfies min(UBΛ_d,UBΛ, PECUBΛ, ICUBΛ, CUB), where the bounds were computed for all three choices $FA1$, $FA2$ and GA of the conditioning variable Λ. In general, we have that partially exact/comonotonic upper bounds (PECUB) are smaller than the improved comonotonic upper bounds (ICUB), which are themselves better than the comonotonic upper bounds (CUB). The detailed numerical results for all bounds are available upon request. Notice that, in general, the Monte Carlo (MC) price is closer to the best lower bound than to the best upper bound. One can also note that the relative distance between the best lower and upper bound is smaller for higher correlation.
We start by discussing Table 1 which corresponds to the case of equal weights, spot prices and volatilities for both assets. In this case the lower bound (24)-(25) applied with Λ given by (14), (16) and (19), which will be denoted by LB$FA1$, LB$FA2$ and

LBGA, are equal. The optimized lower bound LBopt, which is obtained by solving the optimization program (29), gave practically the same values, therefore it is not reported in the table. From all the upper bounds considered, the Rogers and Shi upper bound UBGA_d=LBGA_d+$e^{-rT}\varepsilon(d_{GA})$ (43) with $d = d_{GA}$ (37) based on the geometric average, performs the best.

Table 2 refers to the case of unequal weights and spot prices with equal volatilities. From Table 2, we notice that LB$FA1$ = LB$FA2$ gives sharper results than LBGA. The lower bound LBopt only slightly improves the lower bound LB$FA1$. However, for high volatilities and small ρ, the improvement is significant. As for the upper bounds, we could observe some pattern, namely for out- and at-the-money options, and in-the-money options with the maturity of three years, the Rogers and Shi upper bound UB$FA1_d$=LB$FA1_d$ + $e^{-rT}\varepsilon(d_{FA1})$ performs the best for smaller volatility (0.2), whereas UB$FA2_d$=LB$FA2_d$ + $e^{-rT}\varepsilon(d_{FA2})$ is the best for larger volatility (0.4) with the exception of three years to maturity out-of-the-money option. In the latter case the partially exact/comonotonic upper bound PECUBGA (44)-(46) based on the standardized logarithm of the geometric average outperforms the other bounds for larger volatility. For in-the-money options with the maturity of one year, the pattern is reversed compared to that of in-the-money options with three years to maturity. As mentioned above, we could use the optimization procedure in order to get the best value for ICUB. However, given the experience with the lower bound and the fact that ICUB itself is quite a poor choice for an upper bound, we do not expect the improvement to be so good that it would outperform the best upper bound. Additionally, note that it is possible to compute Rogers and Shi upper bounds based on the optimized values for the lower bound. The results, however, did not outperform the best upper bound.

The second set of input data was taken from Brigo et al. (2002). Here we consider two assets with weights 0.5956 and 0.4044, and spot prices of 26.3 and 42.03, respectively. Maturity is approximately equal to 5 years. The discount factor at payoff is 0.783895779. This example refers to a realistic basket, for which we allow the volatilities and correlations of individual assets to vary in order to facilitate the comparative price analysis. From Table 3 we see that the optimized lower bound gives the best value. The lower bound LB$FA2$ led to the worst results and is therefore not reported. For this example the partially exact/comonotonic upper bound PECUB$FA2$, i.e. with Λ given by $FA2$ (16) turns out to be the sharpest upper bound, except for very high correlation when PECUBGA is to be preferred, and for $\sigma_1 = 0.1$, $\sigma_2 = 0.3$ (for both $\rho = 0.2$ and $\rho = 0.6$) when UBGA_d is the best. As mentioned before for a negative correlation between the assets in the basket the lower bound (24) is not applicable if any of the correlations r_1 and r_2 is negative. If this happens, one should turn to the optimization procedure which enables to choose the coefficients β_1 and β_2 such that r_1 and r_2 would be positive. Consider a case where $\sigma_1 = 0.3$, $\sigma_2 = 0.6$, and $\rho = -0.6$. In this instance we have that the correlations r_1 and r_2 are positive for the conditioning variables $FA1$ and GA and therefore we can find the lower bounds based on those variables: LBFA1= 29.39746493, LBGA= 29.77084284. The optimization procedure (29) gives LBopt = 29.773172314, which shows again that the geometric average is a fairly good choice for a conditioning variable when $a_1 = a_2$, and $S_1(0) = S_2(0)$.

TABLE 1: Comparing bounds, equal weights and spot prices.

	T	corr	vol	MC	LB$FA1$=LB$FA2$=LBGA	UB	Type
$10\%OTM$	1	0.3	0.2	2.90	2.8810	3.2428	UBGA_d
$K = 115.64$			0.4	9.12	9.0280	10.2168	UBGA_d
		0.7	0.2	3.72	3.7172	3.8605	UBGA_d
			0.4	10.88	10.8647	11.3373	UBGA_d
$10\%OTM$	3	0.3	0.2	7.39	7.3290	8.2487	UBGA_d
$K = 127.80$			0.4	18.85	18.4242	21.6818	UBGA_d
		0.7	0.2	8.92	8.9054	9.2733	UBGA_d
			0.4	21.66	21.5913	22.8329	UBGA_d
ATM	1	0.3	0.2	6.44	6.4245	6.6658	UBGA_d
$K = 105.13$			0.4	12.90	12.8088	13.7572	UBGA_d
		0.7	0.2	7.35	7.3445	7.4447	UBGA_d
			0.4	14.64	14.6281	15.0098	UBGA_d
ATM	3	0.3	0.2	11.17	11.1071	11.8210	UBGA_d
$K = 116.18$			0.4	22.41	21.9985	24.8118	UBGA_d
		0.7	0.2	12.69	12.6885	12.9784	UBGA_d
			0.4	25.12	25.0568	26.1324	UBGA_d
$10\%ITM$	1	0.3	0.2	12.37	12.3620	12.4836	UBGA_d
$K = 94.61$			0.4	17.88	17.8093	18.5009	UBGA_d
		0.7	0.2	13.08	13.0861	13.1412	UBGA_d
			0.4	19.47	19.4565	19.7426	UBGA_d
$10\%ITM$	3	0.3	0.2	16.34	16.2843	16.7788	UBGA_d
$K = 104.57$			0.4	26.62	26.2563	28.5970	UBGA_d
		0.7	0.2	17.71	17.6942	17.9022	UBGA_d
			0.4	29.19	29.1130	30.0151	UBGA_d

TABLE 2: Comparing bounds, different weights and spot prices.

	T	corr	vol	MC	LBGA	LB$FA1$ =LB$FA2$	LBopt	UB	Type
$10\%OTM$	1	0.3	0.2	2.57	2.4677	2.5611	2.5611	2.8737	UB$FA1_d$
$K = 101.76$			0.4	8.07	7.7665	7.9855	7.9855	9.0400	UB$FA2_d$
		0.7	0.2	3.28	3.2381	3.2788	3.2788	3.4057	UB$FA1_d$
			0.4	9.61	9.4864	9.5767	9.5767	9.9963	UB$FA2_d$
$10\%OTM$	3	0.3	0.2	6.53	6.2970	6.4823	6.4823	7.3026	UB$FA1_d$
$K = 112.47$			0.4	16.65	15.8604	16.2771	16.2772	18.9776	PECUBGA
		0.7	0.2	7.84	7.7706	7.8478	7.8478	8.1762	UB$FA1_d$
			0.4	19.07	18.8624	19.0234	19.0234	20.0905	PECUBGA
ATM	1	0.3	0.2	5.69	5.5582	5.6750	5.6750	5.8848	UB$FA1_d$
$K = 92.51$			0.4	11.39	11.0722	11.3112	11.3113	12.1387	UB$FA2_d$
		0.7	0.2	6.47	6.4267	6.4724	6.4724	6.5595	UB$FA1_d$
			0.4	12.90	12.7972	12.8889	12.8889	13.2216	UB$FA2_d$
ATM	3	0.3	0.2	9.89	9.6011	9.8066	9.8066	10.4308	UB$FA1_d$
$K = 102.24$			0.4	19.76	18.9795	19.4182	19.4186	21.9157	UB$FA2_d$
		0.7	0.2	11.18	11.0985	11.1778	11.1778	11.4310	UB$FA1_d$
			0.4	22.12	21.9132	22.0729	22.0729	23.0263	UB$FA2_d$
$10\%ITM$	1	0.3	0.2	10.90	10.7924	10.8905	10.8906	10.9984	UB$FA2_d$
$K = 83.26$			0.4	15.78	15.4667	15.7025	15.7027	16.3073	UB$FA1_d$
		0.7	0.2	11.52	11.4815	11.5195	11.5195	11.5680	UB$FA2_d$
			0.4	17.13	17.0467	17.1329	17.1329	17.3822	UB$FA1_d$
$10\%ITM$	3	0.3	0.2	14.41	14.1593	14.3585	14.3586	14.7923	UB$FA1_d$
$K = 92.02$			0.4	23.46	22.7133	23.1587	23.1598	25.2074	UB$FA2_d$
		0.7	0.2	15.58	15.5092	15.5827	15.5827	15.7644	UB$FA1_d$
			0.4	25.69	25.4874	25.6415	25.6416	26.4286	UB$FA2_d$

$n = 2$, $r = 0.05$, K: strike price, MC: Monte Carlo price
Table 1: $a_i = 0.5, i = 1, 2$, $S_i(0) = 100, i = 1, 2$
Table 2: $a_1 = 0.3$, $a_2 = 0.7$, $S_1(0) = 130$, $S_2(0) = 70$
LB$FA1$: lower bound with $\Lambda = \sum_{i=1}^n \beta_i \sigma_i B_i(T)$, $\beta_i = a_i S_i(0) e^{(r-\frac{\sigma_i^2}{2})T}$
LBGA: lower bound with $\Lambda = \sum_{i=1}^n \beta_i \sigma_i B_i(T)$, $\beta_i = a_i / \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j \rho_{ij}}$
LBopt: lower bound obtained via optimization procedure
UB: the smallest value over all upper bounds considered
Type: indicates which upper bound produces the smallest value

TABLE 3: Comparing bounds, different weights and spot prices, different correlations.

Data	σ_1	σ_2	corr	MC	LB$FA1$	LBGA	LBopt	UB	Type
$n = 2$	0.1	0.3	0.2	26.31	26.2206	26.2346	26.2380	26.7612	UBGA_d
$T = 5$ years			0.6	27.48	27.4304	27.4321	27.4356	27.7353	UBGA_d
$K = 32.661$			0.99	28.51	28.5083	28.5083	28.5084	28.5213	PECUBGA
$r = 4.8696\%$	0.1	0.6	0.2	34.15	33.9755	34.0185	34.0233	34.7658	PECUB$FA2$
$a_1 = 0.5956$			0.6	35.64	35.4995	35.5172	35.5206	35.9272	PECUB$FA2$
$a_2 = 0.4044$			0.99	36.85	36.8709	36.8713	36.8714	36.8829	PECUBGA
$S_1(0) = 26.3$	0.3	0.6	0.2	39.92	38.8396	38.9627	38.9640	42.6078	PECUB$FA2$
$S_2(0) = 42.03$			0.6	42.66	42.2886	42.3316	42.3318	43.9641	PECUB$FA2$
			0.99	45.14	45.1919	45.1926	45.1926	45.2273	PECUBGA

6. Asian basket options

An Asian basket option is an option whose payoff depends on an average of values at different dates of a portfolio (or basket) of assets, or which is equivalent on the portfolio value of an average of asset prices taken at different dates. The price of a discrete arithmetic Asian basket call option at current time $t = 0$ is given by

$$ABC(n, K, T) = e^{-rT} E^Q \left[\left(\sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{m-1} b_j S_\ell(T-j) - K \right)_+ \right]$$

with a_ℓ and b_j positive coefficients. For $T \leq m-1$ we call this Asian basket call option in progress and for $T > m-1$, we call it forward starting.
Remark that the double sum $\mathbb{S} = \sum_{\ell=1}^{n} a_\ell \sum_{j=0}^{m-1} b_j S_\ell(T-j)$ is a sum of lognormal distributed variables:

$$\mathbb{S} = \sum_{i=1}^{mn} X_i = \sum_{i=1}^{mn} \alpha_i e^{Y_i}$$

with

$$\alpha_i = a_{\lceil \frac{i}{n} \rceil} b_{i \bmod n-1} S_{\lceil \frac{i}{n} \rceil}(0) e^{(r-\frac{1}{2}\sigma^2_{\lceil \frac{i}{n} \rceil})(T-i \bmod n+1)}$$

and

$$Y_i = \sigma_{\lceil \frac{i}{n} \rceil} W_{\lceil \frac{i}{n} \rceil}(T - i \bmod n + 1) \sim N\left(0, \sigma^2_{Y_i} = \sigma^2_{\lceil \frac{i}{n} \rceil}(T - i \bmod n + 1)\right)$$

for all $i = 1, \ldots, mn$.
Hence, we can apply the general formulae for lognormals from Section 3 (see also Vanmaele et al. (2002)).

7. Conclusion

We derived lower and upper bounds for the price of the arithmetic basket call options using and combining different ideas and techniques such as firstly conditioning on some random variable as in Rogers and Shi (1995), and secondly, results based on comonotonic risks and bounds for stop-loss premiums of sums of dependent random variables as in Kaas, Dhaene and Goovaerts (2000), and finally adaptation of the error bound of Rogers and Shi as in Nielsen and Sandmann (2002). Notice that all bounds have analytical and easily computable expressions. For the numerical illustration it was important to find and motivate a good choice of the conditioning variables appearing in the formulae. We also managed to find the best lower bound through an optimization procedure.

Acknowledgements

G. Deelstra, J. Liinev and M. Vanmaele would like to acknowledge the financial support by the BOF-project 001104599 of the Ghent University.

References

ARTS P. (1999). *Pricing the basket option and exploring its volatility structure.* MSc Thesis, Utrecht University.

BEISSER J. (2001). *Topics in Finance - A conditional expectation approach to value Asian, Basket and Spread Options.* Ph.D. Thesis, Johannes Gutenberg University Mainz.

BRIGO D., MERCURIO F., RAPISPARDA F., AND SCOTTI R. (2002). *Approximated moment-matching dynamics for basket-options simulation.* Working paper, Banca IMI, Milano.

CURRAN, M. (1994). Valuing Asian and portfolio options by conditioning on the geometric mean price. *Management Science*, **40**(12), 1705–1711.

DAHL L.O., AND BENTH F.E. (2001a). Valuation of Asian basket options with quasi-Monte Carlo techniques and singular value decomposition. *Pure Mathematics*, No. 5, February.

DAHL L.O., AND BENTH F.E. (2001b). Fast evaluation of the Asian basket option by singular value decomposition. *Pure Mathematics*, No. 8, March.

DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R. AND VYNCKE, D. (2002). The concept of comonotonicity in actuarial science and finance: theory. *Insurance: Mathematics & Economics*, **31**, 3–33.

KAAS R., DHAENE J., AND GOOVAERTS M.J. (2000). Upper and lower bounds for sums of random variables. *Insurance: Mathematics & Economics*, **27**, 151–168.

NIELSEN, J.A., AND SANDMANN, K. (2002) Pricing bounds on Asian options. Forthcoming in *Journal of Financial and Quantitative Analysis*.

ROGERS L.C.G., AND SHI Z. (1995). The value of an Asian option. *Journal of Applied Probability*, **32**, 1077–1088.

SIMON, S., GOOVAERTS, M.J., AND DHAENE, J. (2000) An easy computable upper bound for the price of an arithmetic Asian option. *Insurance: Mathematics & Economics*, **26**(2-3), 175–184.

VAN DIEPEN E. (2002). *Baskets with a smile. An alternative model for pricing basket options.* MSc Thesis, Free University Amsterdam.

VANMAELE M., DEELSTRA G., LIINEV J., DHAENE J., AND GOOVAERTS M.J. (2002). Bounds for the price of discretely sampled arithmetic Asian options. Submitted.