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MEASUREMENT ERRORS AND ERROR ANALYSIS

Measurement techniques



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METEN IS WETEN, GISSEN IS MISSEN (MEASURING IS KNOWING, GUESSING IS ERRING)

- To measure = to quantify a physical property
 - Aim : understand physical phenomena
 - Method:
 - Determine characteristic properties (T, p, displacement,...)
 - Determine resulting parameters (heat flux, force, ...)
 - Find the relation between the characteristic properties and the resulting parameters (model)
- Simulations : also to quantify a physical property
 - Aim : understand physical phenomena
 - Method :
 - Develop a model based on the conservation laws.



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MEASUREMENTS ARE NOT PRECISE, THERE IS AN ERROR

- Measurement instruments have an ACCURACY or a measurement SCALE
 - A ruler : 1mm
- Result :
 - Measurement errors exist
 - This is not an erroneous measurement, but in inherent inaccuracy of the measurement
 - NOT AVOIDABLE !
 - KEEP THEM AS LOW AS POSSIBLE!
- A measurement reported without error indication and argumentation is worthless and can be misleading



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REPORTING MEASUREMENTS

Measured value of $x = x_{best} \pm error$

- Error are rounded to 1 meaningful figure

$$x = x_{best} \pm \delta x$$

$$x = 8.5342 \pm 0.0234$$

$$x = 8.53 \pm 0.02$$

- Intermediate results use 1 meaningful figure extra

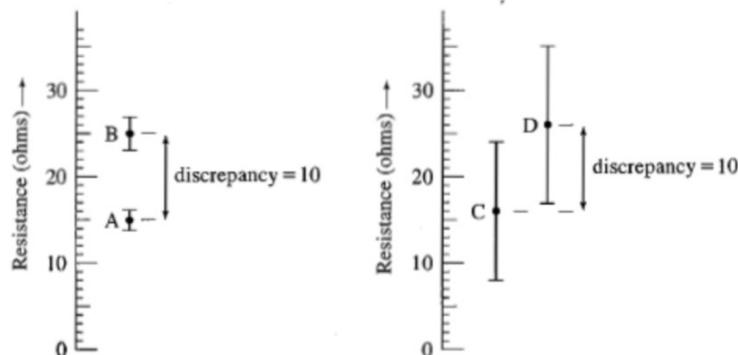
$$x = 8.534 \pm 0.02$$



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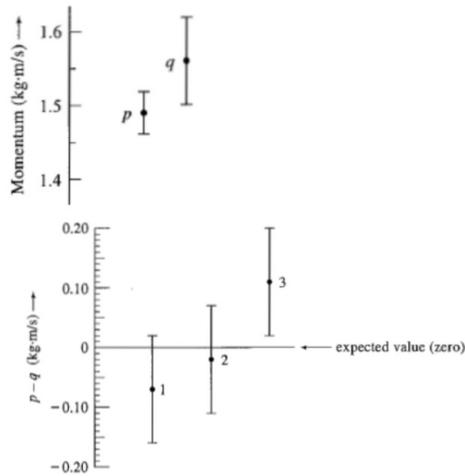
DISCREPANCY BETWEEN MEASUREMENTS

- **Discrepancy** = *difference between the best estimate of two measured values of the same quantity*
- If the discrepancy is bigger than the error margin, then the measurement is invalid



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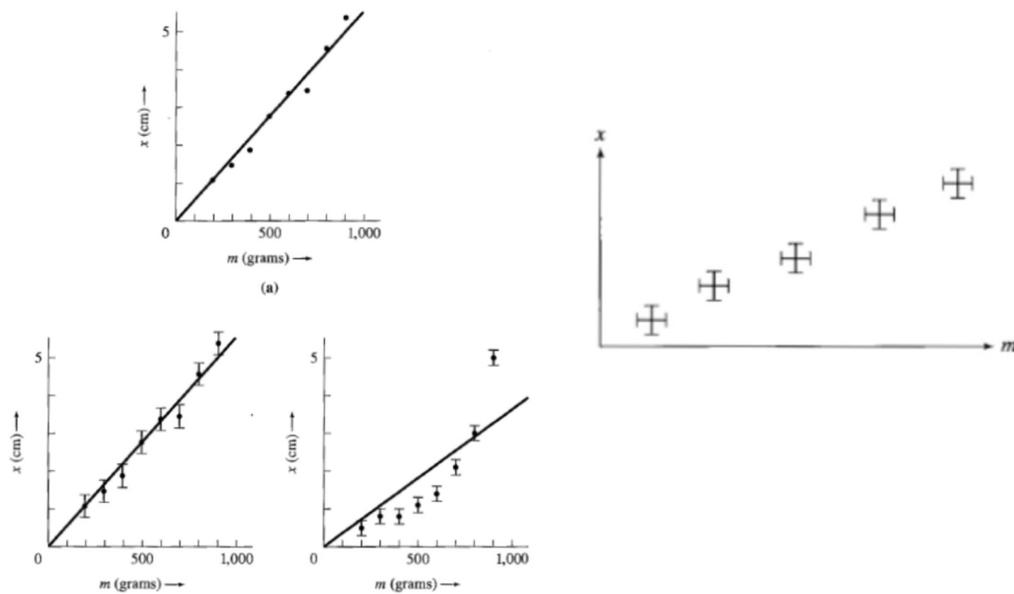
COMPARISON OF TWO MEASUREMENTS



$$\delta(x + y) \approx \delta x + \delta y$$

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GRAFICAL REPRESENTATION AND CURVE FITTING



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ABSOLUTE EN RELATIVE ERROR

- Absolute error

$$x = x_{best} \pm \delta x$$

- Rounding of rel error:

- 1 figure : 10% tot 100%

- 2 figures : 1% tot 10 %

- 3 figures : 0.1 tot 1%

$$rel\ error = \pm \frac{\delta x}{|x_{best}|}$$

- Error of a product:

(with small rel errors)

$$q = xy$$

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$



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ERROR TYPES

- How do we determine errors?

- 1) all measurement equipment have an accuracy. There is a relation between the measured quantity and the measurement equipment. This is called ERROR ANALYSIS.

- 2) Repeat the measurements and do a statistical analysis

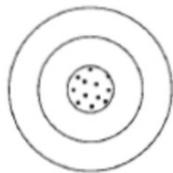
- Random errors : statistical analysis

- Systematic errors : not statistical



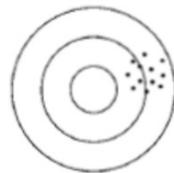
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EXAMPLE



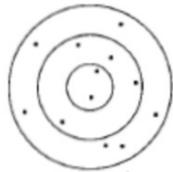
Random: small
Systematic: small

(a)



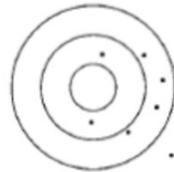
Random: small
Systematic: large

(b)



Random: large
Systematic: small

(c)



Random: large
Systematic: large

(d)



Random: small
Systematic: ?

(a)



Random: small
Systematic: ?

(b)



Random: large
Systematic: ?

(c)



Random: large
Systematic: ?

(d)

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STATISTICAL CONSIDERATIONS

– Average value

$$\bar{x} = \frac{\sum_{j=1..N} x_j}{N}$$

– Deviation

$$d_j = x_j - \bar{x}$$

$$\begin{aligned} \sum d_j &= \sum x_j - \sum \bar{x} \\ &= N\bar{x} - N\bar{x} \\ &= 0 \end{aligned}$$

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STATISTICAL CONSIDERATIONS

- Standard deviation

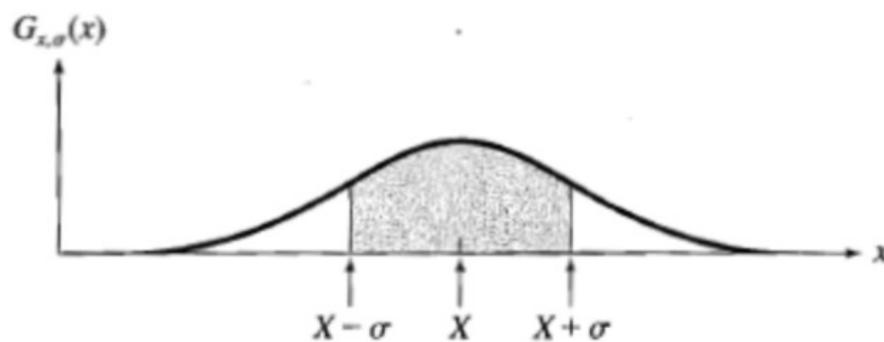
$$\sigma_x = \sqrt{\frac{1}{N-1} \sum d_j^2} = \sqrt{\frac{1}{N-1} \sum (x_j - \bar{x})^2}$$



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GAUSS NORMAL DISTRIBUTION

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$



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WHAT IS A GOOD ESTIMATE OF X AND σ ?

- Consider N measurements of x_j
- What is the chance we measure x_j :

$$\text{Probability}(x_j) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_j-X)^2}{2\sigma^2}}$$

$$\text{Prob}(x_1, x_2, \dots, x_j, \dots, x_N) = \text{Prob}(x_1) \times \text{Prob}(x_2) \times \dots \times \text{Prob}(x_N)$$

$$\propto \frac{1}{\sigma^N} e^{-\sum \frac{(x_j-X)^2}{2\sigma^2}}$$



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WHAT IS A GOOD ESTIMATE OF X AND σ ?

- When is the **probability the biggest** that X and σ are the result of a set of measurements x_j ?

- Maximize $\text{Prob}(x_1, x_2, \dots, x_j, \dots, x_N) \propto \frac{1}{\sigma^N} e^{-\sum \frac{(x_j-X)^2}{2\sigma^2}}$

- Minimize $\sum \frac{(x_j-X)^2}{2\sigma^2}$

$$\sum (x_j - X) = 0$$

$$\sum x_j - \sum X = \sum x_j - NX = 0$$

$$X = \frac{\sum x_j}{N}$$



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WHAT IS A GOOD ESTIMATE OF X AND σ ?

- You can also derivate :

$$\sigma = \sqrt{\frac{1}{N-1} \sum (x_j - \bar{x})^2}$$



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WHAT IS THE ERROR ON X ?

- What is the meaning of σ
- In a Gauss normal distribution 68% of all results lie within the interval

$$x = \bar{x} \pm \sigma_x$$

- For 1 measurement x_j
- $$x = x_j \pm \delta x$$

$$\delta x = \sigma_x \text{ with } 68\% \text{ confidence interval}$$

- For N measurements :

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$



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ERROR PROPAGATION

- Different measured values $x, y, z \dots$
- A resulting value is $q=f(x,y,z,\dots)$
- Errors on x,y,z are $\delta x, \delta y, \delta z$
- **What is the error on q : δq ?**

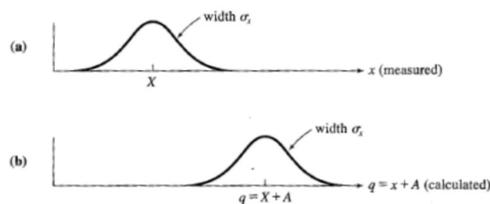


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SIMPLE CASE 1 : ADD A CONSTANT A

$$q = x + A$$

$$\text{Prob}(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}}$$



$$\text{Prob}(q) = \text{Prob}(q - A) = \text{Prob}(x) \propto e^{-\frac{(q-A-X)^2}{2\sigma_x^2}}$$

$$\text{Prob}(q) \propto e^{-\frac{(q-(X+A))^2}{2\sigma_x^2}}$$



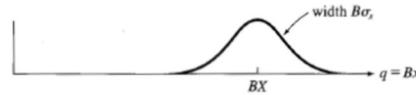
– The width of the distribution stays the same, so does the error !

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SIMPLE CASE 2 : MULTIPLY WITH A CONSTANT

$$q = Bx$$

$$\text{Prob}(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}}$$



$$\begin{aligned} \text{Prob}(q) = \text{Prob}(q/B) &\propto e^{-\frac{(q/B-X)^2}{2\sigma_x^2}} \\ &= e^{-\frac{(q-BX)^2}{2B^2\sigma_x^2}} \end{aligned}$$

– The width of the distribution is changed to $B \sigma_x$!

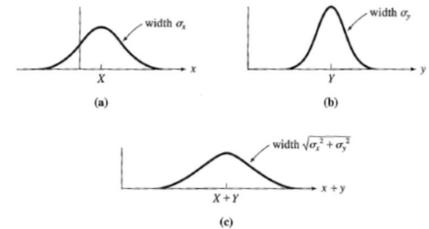


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THE ADDITION OF TWO VALUES

$$q = x + y$$

$$\text{Prob}(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}} \quad \text{Prob}(y) \propto e^{-\frac{(y-Y)^2}{2\sigma_y^2}}$$



– Als nu $X=Y=0$

$$\text{Prob}(x, y) \propto e^{-\frac{x^2}{2\sigma_x^2}} e^{-\frac{y^2}{2\sigma_y^2}} = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = \frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + z^2$$

$$\text{Prob}(x, y) \propto e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + z^2\right)} = e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2}\right)} e^{-\frac{1}{2}z^2} = \text{Prob}(x+y, z)$$



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THE ADDITION OF TWO VALUES

$$\text{Prob}(x+y) \propto \int_{-\infty}^{+\infty} \text{Prob}(x+y, z) dz$$

$$\text{Prob}(x+y, z) = e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2}\right)} e^{-\frac{1}{2}(z^2)}$$

$$\text{Prob}(x+y) \propto e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2}\right)}$$

$$\sigma_{x+y} = \sqrt{\sigma_x^2 + \sigma_y^2}$$



$$q = x + y = (x - X) + (y - Y) + (X + Y)$$

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IN GENERAL

$$f(x+u, y+v) = f(x, y) + \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v$$

$$q = q(x, y) = q(X, Y) + \frac{\partial q}{\partial x} (x - X) + \frac{\partial q}{\partial y} (y - Y)$$

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x} \sigma_x\right)^2 + \left(\frac{\partial q}{\partial y} \sigma_y\right)^2}$$



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WHAT IS THE ERROR ON THE AVERAGE VALUE?

- For N measurements x_1, \dots, x_N :

$$\bar{x} = \frac{\sum_{j=1..N} x_j}{N}$$

- Do a lot of experiments containing N measurements
- Error propagation :

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{\partial \bar{x}}{\partial x_1} \sigma_{x_1}\right)^2 + \dots + \left(\frac{\partial \bar{x}}{\partial x_N} \sigma_{x_N}\right)^2}$$

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N} \sigma_x\right)^2 + \dots + \left(\frac{1}{N} \sigma_x\right)^2}$$



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REPRESENTATION OF MEASURED VALUES

- 68% of the measured values lie within the interval

$$\begin{aligned} x &= \bar{x} \pm \sigma_{\bar{x}} \\ &= \bar{x} \pm \frac{\sigma_x}{\sqrt{N}} \end{aligned}$$

- 95% of the measured values lie within the interval

$$\begin{aligned} x &= \bar{x} \pm 2\sigma_{\bar{x}} \\ &= \bar{x} \pm 2 \frac{\sigma_x}{\sqrt{N}} \end{aligned}$$



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DISCARDION OF MEASUREMENTS

- Assume 1 measured value seems not to lie within the normal/expected value range. What to do with it?
- 1) check whether something went wrong during the measurement . If so, discard.
- 2) If there is no reason to assume there was a mistake: use the : Chauvenet criterion

$$\text{Pr ob}(\text{outside } \sigma_x) \quad t = \frac{|x_{\text{verd}} - \bar{x}|}{\sigma_x}$$

- Discard if :

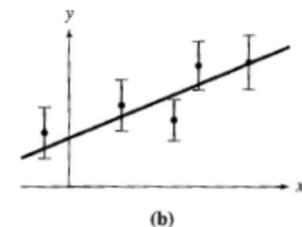
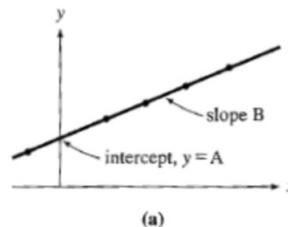
$$n = N \times \text{Pr ob}(\text{outside } \sigma_x) < 0.5$$



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LEAST SQUARE DATA FITTING

$$y = Ax + B$$



$$\sigma_y = \sqrt{\frac{1}{N-2} \sum (y_j - A - Bx_j)^2}$$

$$\sigma_A = \sigma_y \sqrt{\frac{\sum (x_j)^2}{\Delta}} \quad \sigma_B = \sigma_y \sqrt{\frac{N}{\Delta}} \quad \Delta = N \sum x^2 - (\sum x)^2$$



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CONCLUSION

- Measurements contain errors
- Measurement equipment has an accuracy : look to the manual
- Repeated measurements : statistical analysis
- Error analysis :
 - Estimate the error on the result
 - Discuss the error analysis :
 - Measurement accuracies
 - Calculate the error propagation
- **FIRST PREFORM AN ERROR ANALYSIS and DESIGN the experiment to reduce errors**



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