The asymptotic volume of the Birkhoff polytope

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Doubly-stochastic matrices

A doubly-stochastic matrix has non-negative real entries and all row sums and column sums equal to 1.

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

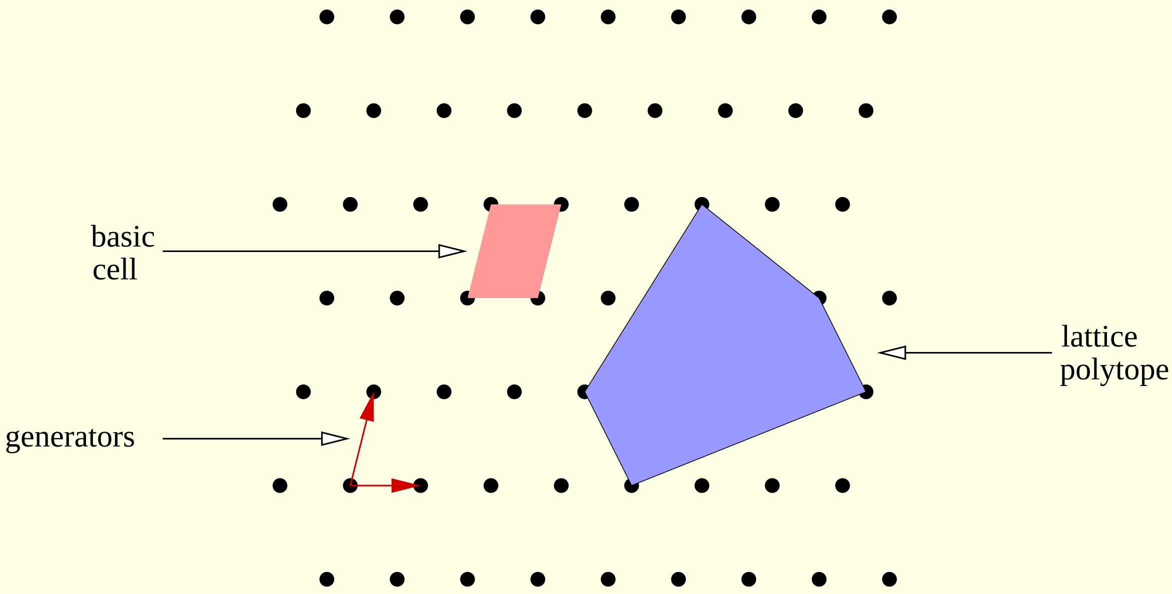
The Birkhoff–von Neumann polytope $\mathcal{B}_n$ is the set of all $n \times n$ doubly-stochastic matrices.

Birkhoff–von Neumann Theorem. $\mathcal{B}_n$ is a polytope whose extreme points (vertices) are the permutation matrices.

Lattice polytopes

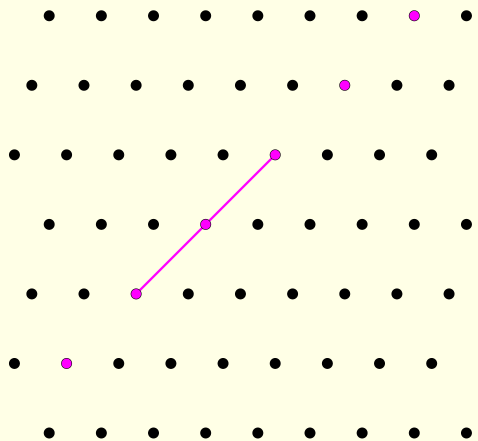
A lattice is the set of integer combinations of some set of generators.

A lattice polytope is the closed convex hull of some set of lattice points.



Volume (area) can be measured in the standard fashion, or in terms of the basic cell. The latter is the relative volume.

A lattice polytope can have dimension lower than the lattice. The volume and relative volume are measured according to the span of the polytope and the sublattice it induces.



Volume = the length of the line segment

Relative volume = 2

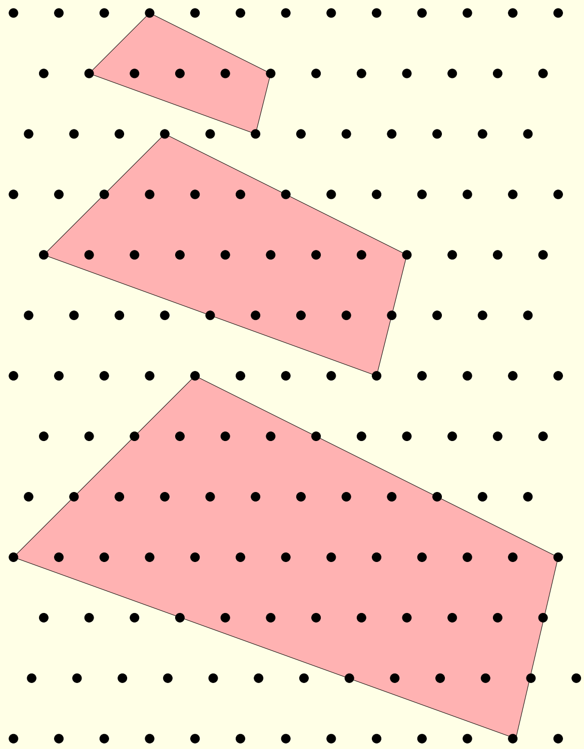
Ehrhart polynomials

For lattice polytope $\mathcal{P}$, let $E(\mathcal{P}, t)$ = the number of lattice points in $t\mathcal{P}$.

$E(\mathcal{P}, 1) = 7$

$E(\mathcal{P}, 2) = 21$

$E(\mathcal{P}, 3) = 43$



A celebrated theorem of Ehrhart says that $E(\mathcal{P}, t)$ is a polynomial for integer t.

For the example:

$$E(\mathcal{P}, t) = 4t^2 + 2t + 1$$

For large t, the relative volume of $t\mathcal{P}$ is approximately $E(\mathcal{P}, t)$ (boundary effects become negligible).

Therefore:

The leading coefficient of $E(\mathcal{P}, t)$ is the relative volume.

Therefore:

If d is the dimension, the relative volume is

$$\lim_{t \to \infty} E(\mathcal{P}, t)/t^d.$$

Birkhoff polytopes

$\mathcal{B}_n$ is a subset of $\mathbb{R}^{n\times n}$, but has dimension $(n-1)^2$.

$$\begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{pmatrix}$$

Using the lattice $\mathbb{Z}^{n\times n}$,

$$E(\mathcal{B}_1, t) = 1$$

$$E(\mathcal{B}_2, t) = t + 1$$

$$E(\mathcal{B}_3, t) = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$$

Beck and Pixton (2006) found $(\mathcal{B}_n, t)$ for $t \leq 9$ and the leading term for $t = 10$.

The lattice points in $t\mathcal{B}_n$ are the $n \times n$ matrices of non-negative integers whose row and columns are all equal to t.

A lattice point in $6\mathcal{B}_3$:

$$\begin{pmatrix} 1 & 0 & 5 \\ 4 & 2 & 0 \\ 1 & 4 & 1 \end{pmatrix}$$

The volume of a basic cell is n^{n-1} (Diaconis and Efron, 1985).

Therefore, the volume of $\mathcal{B}_n$ is

$$n^{n-1} \lim_{t \to \infty} \frac{M(n,t)}{t^{(n-1)^2}},$$

where $M(n,t)$ is the number of $n \times n$ matrices of non-negative integers whose row and columns are all equal to t.

Integer matrices

Let $M(m, s; n, t)$ be the number of non-negative integer matrices of order $m \times n$, such that each row sum is s and each column sum is t.

Obviously we need $ms = nt$.

An example of a matrix counted by $M(3, 20; 4, 15) = 965071$:

$$\begin{array}{cccc|c} 5 & 4 & 0 & 11 & 20 \\ 2 & 8 & 9 & 1 & 20 \\ 8 & 3 & 6 & 3 & 20 \\ \hline 15 & 15 & 15 & 15 & \end{array}$$

$M(18, 10; 20, 9) =$ 717,197,213,652,153,417,464,506,119,278,691,
437,121,387,356,187,926,222,586,832,234,613,738,328,
256,525,810,124,867,047,876,056,407,343,493,714,445,200

Asymptotic value of $M(m, s; n, t)$?

We seek the asymptotic value of $M(m, s; n, t)$ as $m, n \to \infty$ with $s = s(m, n)$, $t = t(m, n)$.

Despite a large literature on $M(m, s; n, t)$, there are very few exact or asymptotically exact results.

The case $s, t = O(1)$ was solved by Everett and Stein (1971), Békéssy, Békéssy and Komlós (1972), and Bender (1974).

The case $st = o\big((mn)^{1/2}\big)$ was solved by Greenhill and McKay (2007).

However, these results do not help with $\mathcal{B}_n$, since that requires $s, t \to \infty$ for fixed m, n.

Algebraic approach

$M(m, s; n, t)$ is the coefficient of $x_1^s x_2^s \cdots x_m^s \, y_1^t y_2^t \cdots y_n^t$ in

$$F(x_1, \ldots, x_m, y_1, \ldots, y_n) = \prod_{j=1}^{m} \prod_{k=1}^{n} \bigl(1 + x_j y_k + x_j^2 y_k^2 + x_j^3 y_k^3 + \cdots\bigr)$$

$$= \prod_{j=1}^{m} \prod_{k=1}^{n} \bigl(1 - x_j y_k\bigr)^{-1}$$

Define the density $\lambda = s/n = t/m$, which is the average value of a matrix entry.

We now estimate the required coefficient by the saddle-point method in n complex dimensions.

Integrate each variable around a circle of radius

$$\sqrt{\frac{\lambda}{1+\lambda}}\,.$$

Algebraic approach (continued)

$$x_j = \sqrt{\frac{\lambda}{1+\lambda}}\, e^{i\theta_j}, \qquad y_k = \sqrt{\frac{\lambda}{1+\lambda}}\, e^{i\phi_k}.$$

By Cauchy's theorem,

$$M(m,s;n,t) = \frac{1}{(2\pi)^{m+n}} \big(\lambda^{-\lambda}(1+\lambda)^{1+\lambda}\big)^{mn}\, I(m,n),$$

where

$$I(m,n) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j,k}\big(1-\lambda(e^{i(\theta_j+\phi_k)}-1)\big)^{-1}}{\exp\big(is\sum_j \theta_j + it\sum_k \phi_k\big)}\, d\theta_1 \cdots d\theta_m\, d\phi_1 \cdots d\phi_n\,.$$

The integrand has an obvious symmetry:

Add a constant to each θ_j and subtract the same constant from each ϕ_k.

Estimation of the integral

1. Guess that the value comes mostly from a small region near where the integrand is maximum.

2. Inside the central region:

   (a) Change variables down to dimension $m + n - 1$.

   (b) Expand the integrand up to terms of order 4.

   (c) Infer the integral over the central region.

3. Bound the integral outside the central region.

Analytic difficulties require us to limit how non-square the matrix can be and what values λ can take:

$$n = O(m \log m) \text{ and } m = O(n \log n),$$
$$1/\log n \le \lambda \le n^{O(1)}.$$

The integral within the central region

The central region is where $\theta_j + \phi_k$ is close to 0 for all j, k. This implies that the θ_j's are approximately equal to some value and the ϕ_k's are approximately equal to the negative of that value.

Specifically, we define

$$\bar{\boldsymbol{\theta}} = \frac{1}{m}\sum_{j=1}^{m} \theta_j$$

$$\bar{\boldsymbol{\phi}} = \frac{1}{n}\sum_{k=1}^{n} \phi_k$$

then we change variables as follows:

$$\begin{aligned} \nu &= \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\phi}} \\ \mu &= \bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\phi}} \\ \hat{\theta}_j &= \theta_j - \bar{\boldsymbol{\theta}} \qquad (1 \le j \le m{-}1) \\ \hat{\phi}_k &= \phi_k - \bar{\boldsymbol{\phi}} \qquad (1 \le k \le n{-}1) \end{aligned}$$

The integrand is independent of ν, so we remove it.

The other variables define the central region:

$$\text{Central region}\ :\qquad \begin{aligned} |\mu| &\le (1+\lambda)^{-1}(mn)^{-1/2+2\epsilon} \\ |\hat\theta_j| &\le (1+\lambda)^{-1} n^{-1/2+\epsilon}, \quad 1 \le j \le m \\ |\hat\phi_k| &\le (1+\lambda)^{-1} m^{-1/2+\epsilon}, \quad 1 \le k \le n \end{aligned}$$

Within the central region, the integrand can be approximated by its Taylor expansion to order 4:

$$\begin{aligned} \exp\Big(&-\tfrac{1}{2}\,\lambda(1+\lambda)\sum_{j,k}(\mu+\hat\theta_j+\hat\phi_k)^2 \\ &-\tfrac{i}{6}\,\lambda(1+\lambda)(1+2\lambda)\sum_{j,k}(\mu+\hat\theta_j+\hat\phi_k)^3 \\ &+\tfrac{1}{24}\,\lambda(1+\lambda)(1+6\lambda+6\lambda^2)\sum_{j,k}(\mu+\hat\theta_j+\hat\phi_k)^4 + o(1)\Big) \end{aligned}$$

Finishing the integral within the central region

Next we apply the linear transformation that diagonalises the quadratic form.

Unfortunately this has two unpleasant consequences:

(a) The region of integration is no longer a cuboid, but close to the intersection of a cuboid and the region between two parallel planes.

(b) The cubic and quartic terms are messed up badly.

These problems can be handled, and the integration can be performed.

The integral outside the central region

The part of the region of integration outside the central region contributes negligibly.

This can be shown by breaking the region into a large number of pieces and bounding the integrand in each one.

The details are very messy.

Very dense matrices

To estimate the volume of $\mathcal{B}_n$ we we need the behaviour of $M(n,t) = M(n,t;n,t)$ as $t \to \infty$ for fixed n.

However, our analysis so far required $t = n^{O(1)}$.

A theorem of Richard Stanley comes to the rescue:

If $\mathcal{P}$ is a lattice polytope of dimension d, then there are nonnegative $h_0,\ldots,h_d$ such that for integer $t \geq 1$,

$$E(\mathcal{P},t) = \sum_{i=0}^{d} h_{d-i} \binom{t+i}{d}.$$

This is enough to show that for $\alpha \geq n^5$,

$$\frac{M(m,\alpha n; n, \alpha m)}{M(m, n^6; n, mn^5)} = \left(\frac{\alpha}{n^5}\right)^{(m-1)(n-1)} \bigl(1+o(1)\bigr).$$

Asymptotic value of $M(m, s; n, t)$

Recall:

The number of ways to write n as a sum of k nonnegative terms is

$$\binom{n+k-1}{k-1}.$$

1. The number of $m \times n$ matrices whose overall sum is $ms = nt$ is

$$N = \binom{mn+ms-1}{mn-1}.$$

Now choose a random matrix with this sum.

2. The probability that the row sums all equal s (Event R) is

$$P_1 = \binom{n+s-1}{n-1}^m \Big/ N.$$

Asymptotic value of $M(m, s; n, t)$ (continued)

3. The probability that the column sums all equal t (Event C) is

$$P_2 = \binom{m+t-1}{m-1}^n \Big/ N.$$

4. If events R and C were independent, we would have

$$M(m, s; n, t) = N\, P_1\, P_2.$$

In fact we have

$$M(m, s; n, t) = N\, P_1\, P_2 \exp\Big(\tfrac{1}{2} + o(1)\Big).$$

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Interesting fact: The same formula holds in the sparse case.

Volume of the Birkhoff polytope

For any $\varepsilon > 0$,

$$\mathrm{vol}(\mathcal{B}_n) = \frac{1}{(2\pi)^{n-1/2} n^{(n-1)^2}} \exp\Big(n^2 + \frac{1}{3} + O(n^{-1/2+\varepsilon}) \Big)$$

as $n \to \infty$.

n	estimate/actual
1	1.51345
2	1.20951
3	1.25408
4	1.22556
5	1.19608
6	1.17258
7	1.15403
8	1.13910
9	1.12684
10	1.11627

What's next?

The mysterious constant $e^{1/2}$ has been proved for:

(i) $st = o\big((mn)^{1/2}\big)$ (Greenhill & McKay)

(ii) $m \approx n$, $s, t \geq 1/\log n$ (Canfield & McKay)

Supported by a large number of exact values, we conjecture

$$M(m, s; n, t) = N\, P_1\, P_2 \exp\Big(\tfrac{1}{2} + o(1)\Big).$$

as $m, n \to \infty$ for *any* functions $s = s(m, n)$, $t = t(m, n)$ with $ms = nt$.

Two techniques now exist which probably have the power to prove it:

(a) A recurrence approach by Liebenau and Wormald.

(b) A cumulant expansion approach by Isaev and McKay.